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# METHODS AND APPLICATIONS OF MAX-PLUS ALGEBRAS

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**Abstract**: Exotic algebras such as the max-plus semiring, or the tropical semirings, have been invented many times in various fields: discrete event system theory, communication networks as well as timing analysis of digital circuits. Despite this apparent profusion, there is a set of common basic results and problems, which seems to be useful in most applications. In recent research the relevancy of discrete event systems (DES) steadily increases. In order to handle DES the introduction of a DES-system theory is useful. This article outlines the DES-system theory and shows an algorithm for an efficient calculation of the optimal control in dioids.

## 1. Introduction

Tropical algebra, also called “min-plus” or “max-plus” algebra, is a relatively new topic in mathematics that has recently caught the interest of algebraic geometers, computer scientists, combinatorists, and other mathematicians. According to Andreas Gathmann, tropical algebra was pioneered by mathematician and computer scientist Imre Simon in the 1980s but did not receive widespread attention until a few years ago. Interestingly, the title “tropical algebra” is nothing more than a reference to Simon’s home country - Brazil.

The tropical semi-ring, which is the basis of this problem-solving tool and our major object of study, is formed by replacing the standard addition operation with the minimum function and replacing multiplication with addition. Although many difficult problems can be translated into simpler tropical problems, even these simpler problems can be difficult if we do not understand the tropical algebraic system. Since tropical algebra is so new, several things about it are unknown. Our purpose in studying it, then, is to understand it better, since the more we understand about the mechanics of the tropical algebraic system, the more effective this tool becomes.

Due to the fact that many interesting results in common system theory are based on adapter mathematical description of the system, the development of a DES-system theory was mandatory. After the definition of the corresponding terms the problem of an efficient function calculation arises. This article presents an effective way for the calculation of the star-operator. As already mentioned, an effective computation of the star-operator may be necessary in order to determine the transfer function of a given DES.

The basics of dioid theory are given and applied to DES-system theory. The need of an effective calculation of the star-operator is motivated. As central result a theorem is proved, which allows the solution of an affine equation being slightly less elaborate than matrix multiplication. Finally the relative effort in comparison to the multiplication is visualized.

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## 2. Basics of Dioid-Theory

This section will introduce the definition of a dioid. The considerations in this section will not present proofs but restrict to simple properties. This is done due to the fact that only basic algebraic knowledge is necessary in order to understand the following simplifications. Detailed examinations and characterizations can be found in [1], [2], [3] and [4]. First the algebraic properties of a dioid are defined:

### 2.1 Definition *(Dioid)*

*A dioid is a set $\mathcal{D}$ endowed with two inner operations $\oplus$ and $\otimes$, called addition and multiplication, such that $\oplus$ is associative, commutative, idempotent and has a zero (denoted as $\varepsilon$); $\otimes$ is associative and has a unit (denoted as $e$); $\varepsilon$ is absorbing for $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$) and $\otimes$ distributes over $\oplus$. In most cases the multiplication sign is omitted.*

A matrix dioid $\mathcal{D}^{N\times N}$ is defined on grounds of a dioid $\mathcal{D}$ by:

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{i,k} = \boldsymbol{A}_{i,k} \oplus \boldsymbol{B}_{i,k} \qquad i,k = 1, \dots, N \tag{1}$$

$$(\boldsymbol{A} \otimes \boldsymbol{B})_{i,k} = \bigoplus_{\nu=1}^{N} \left(\boldsymbol{A}_{i,\nu} \otimes \boldsymbol{B}_{\nu,k}\right) \qquad i,k = 1, \dots, N \tag{2}$$

In [4], [5] it is shown that an algebraic structure possessing an idempotent, associative and commutative inner operation can be furnished with an order relation. The induced order $\preccurlyeq$ is given by $a \preccurlyeq b \Leftrightarrow a \oplus b = b \quad a,b \in \mathcal{D}$. Such an ordered set may be completed adding the upper bound of each subset, resulting in a complete dioid:

### *2.2* Definition *(Complete dioid)*

*A dioid is complete, if it is closed for upper bounds*

$$a_i \in \mathcal{D}, i \in I \Rightarrow \bigoplus a_i \in \mathcal{D}, \tag{3}$$

*and the distributivity extends to infinite sums. Sums with an uncountable amount of elements are well defined as upper bounds of certain subsets.*

In complete dioids the solution of an affine equation can be obtained by algebraic calculations. For this purpose the star-operation $*$ will be useful. Therefore an element of $\mathcal{D}$ to the power of *n* is recursively defined by

$$a^0 = e \quad , \quad a^1 = a \quad , \quad a^n = a^{n-1} \otimes a. \tag{4}$$

Through this the star-operation

$$a^* = \bigoplus_{n=0}^{\infty} a^n \tag{5}$$

is defined, which allows the characterization of solutions of affine equations:

### 2.3 Theorem

*In a complete dioid the following statements concerning affine equation $x = ax \oplus b$ hold true:*

1. *$a^* \otimes b$ is the least solution of $x = ax \oplus b$.*
2. *For each solution $\tilde{x}$, the property $\tilde{x} = a^*\tilde{x}$ is valid.*

## 3. Affine Equations in the DES-Theory

This section justifies the necessity of an efficient calculation of the star-operator. For this purpose, results are presented demonstrating the description of DES by means of algebraic methods. For the sake of simplicity these results are not derived in a closed manner, but merely stated as facts. The justification can be found in [1], [5] or [4].

In [4] it has been shown, that synchronization-graphs, a special kind of petri-nets, can be described by state space equations:

$$\boldsymbol{x}(k) = \boldsymbol{A}\boldsymbol{x}(k-1) \oplus \boldsymbol{B}\boldsymbol{u}(k),$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) \oplus \boldsymbol{D}\boldsymbol{u}(k).$$

Hereby the state variables $\boldsymbol{x}_i(k)$ represent the instant in time at which the $i$-th transition of the petri-net is fired for the $k$-th time.

With the $\gamma$-transformation of an signal $\boldsymbol{x}(k)$ as an analogy to the z-transform,

$$\Gamma\{\boldsymbol{x}(k)\} = \bigoplus \boldsymbol{x}(k)\gamma^k, k \in \mathbb{Z}, \tag{6}$$

signals are transformed into an image domain. The delay of a signal corresponds to the multiplication with the formal operator $\gamma$ in the image domain:

$$\Gamma\{\boldsymbol{x}(k)\} = \boldsymbol{X}(\gamma) \Rightarrow \Gamma\{\boldsymbol{x}(k-1)\} = \gamma\boldsymbol{X}(\gamma). \tag{7}$$

In terms of this image domain calculations are simplified and the state space equations become

$$\boldsymbol{X}(\gamma) = \boldsymbol{A}\gamma\boldsymbol{X}(\gamma) \oplus \boldsymbol{B}\boldsymbol{U}(\gamma),$$

$$\boldsymbol{Y}(\gamma) = \boldsymbol{C}\boldsymbol{X}(\gamma) \oplus \boldsymbol{D}\boldsymbol{U}(\gamma).$$

Applying theorem 2.3 the resulting input-output-behavior is given by:

$$\boldsymbol{Y}(\gamma) = (\boldsymbol{C}[\boldsymbol{A}\gamma]^*\boldsymbol{B} \oplus \boldsymbol{D})\boldsymbol{U}(\gamma). \tag{8}$$

Therefore the input-output behavior of the system is determined, calculating the right hand bracket. This computation requires knowledge of the matrix $[\boldsymbol{A}\gamma]^*$ and therefore an effective calculation of this term is essential.

## 4. Calculation of the Star-Operator

As proved in [2], the calculation of the star-operation for a $2 \times 2$-matrix can be reduced to manipulation of the (scalar) matrix coefficients:

### 4.1 Theorem

*In $\mathcal{D}^{2\times2}$ the following calculation of the star-operation takes place:*

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{M}^* = \begin{pmatrix} a^* \oplus a^*b(ca^*b \oplus d)^*ca^* & a^*b(ca^*b \oplus d)^* \\ (ca^*b \oplus d)^*ca^* & (ca^*b \oplus d)^* \end{pmatrix}.$$

At first sight this solution seems quite circumstantial. In fact, the calculation is significantly simplified. Considering the terms of matrix $M^*$ it becomes obvious that the elements can be recursively computed using the similarities between the elements.

### 4.2 Theorem

*The star-operation for a 2 $\times$ 2-matrix can be calculated within two scalar additions, six scalar multiplications and 2 scalar star-operations.*

Using the latter theorem, the computation of matrices of higher dimensions is ascribed to sub-matrices of the half dimension. This is somehow similar to the derivation of the FFT out of the DFT. For this purpose the similarity between two matrix dioids is needed:

### 4.3 Theorem

*Dioids $\mathcal{D}^{2n\times 2n}$ and $(\mathcal{D}^{n\times n})^{2\times 2}$ are isomorph.*

Due to this theorem the calculation of matrices of higher dimensions is reduced to calculations with matrices of smaller dimensions.

### 4.4 Theorem

*For $M \in \mathcal{D}^{n\times n}, n = 2^k$, the star-operation can be calculated with less effort than a matrix multiplication of the same dimension.*

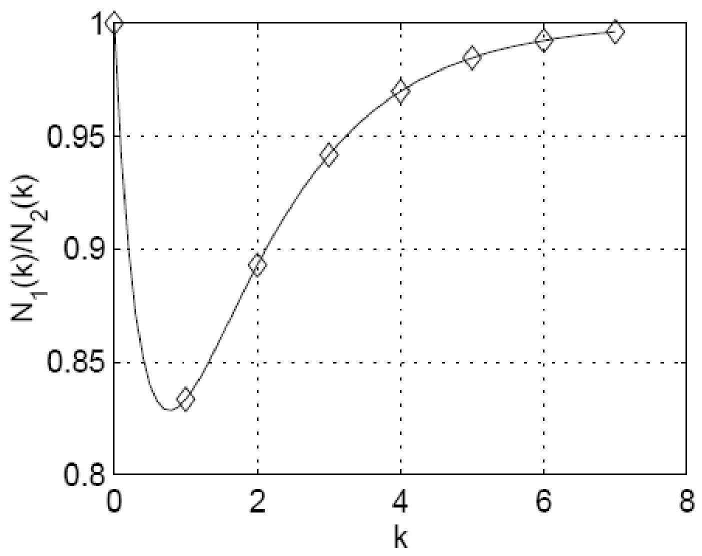


***Figure 1:*** *Relative effort for the calculation of the star operation*

## 5. Application to transportation networks

We are interested in transportation systems and more particularly in bus networks. At first a max-plus linear model of these systems is proposed. The behavior of such systems is controlled by a timetable. Timetables settings are a part of an optimization. A bus network can be modeled as a state representation in $\mathbb{Z}_{max} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as sum and the classical sum as product with the greatest element $+\infty$. Then the resulting dioid is complete.

A transportation network can be modeled by:

$$\boldsymbol{x}(k) = \boldsymbol{A}\boldsymbol{x}(k-1) \oplus \boldsymbol{B}\boldsymbol{u}(k),$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k),$$

in which $\boldsymbol{x}(k)$ is a vector. Matrix $\boldsymbol{A}$ is defined such as $\boldsymbol{A}_{ij} = \boldsymbol{a}_{ij}$ where $\boldsymbol{a}_{ij}$ corresponds to the travelling time from stop *j* to stop *i*.

The timetable is represented by input vector $\boldsymbol{u}(k)$ and entries of matrix $\boldsymbol{B}$ are such as $\boldsymbol{B}_{ij} = e$ if timetable must be respected at *i*, $\boldsymbol{B}_{ij} = \varepsilon$ otherwise. We consider constraints which can be formulated as an implicit inequality over state vector $\boldsymbol{x}$. A relevant goal for the control is to delay as much as possible the input events, i.e. to compute the greatest control vector $\boldsymbol{u}$. Then the synthesis of the optimal control can be formulated as the computation of the greatest fixed point of a mapping $f$ by the following iterative computation:

$$\boldsymbol{u}_0 = -\infty, \;\; \boldsymbol{u}_{k+1} = f(\boldsymbol{u}_k).$$

It can be proved that this computation converges in a finite number $k = N$ of iterations and $\boldsymbol{u}_N$ is the optimal control, i.e. the greatest solution of $f$ with $\boldsymbol{u} \preccurlyeq f(\boldsymbol{u})$.

The iterative computation has been implemented with the C++ library libminmaxgd in [5].

## 6. Summary

This article presents the basics of an efficient calculation for the star-operation in arbitrary dioids. After the motivation of the need of such considerations the effort is determined by a given algorithm for the simplification of the computations. As central theorem of this article it can be proofed that the relative effort of a star-operator is less than the multiplication of matrices of the same dimension. As a consequence the use of the star-operator is no time-critical part of the determination of a DES's behavior.

We have presented a new method to compute a control problem in max-plus linear system theory. Using results on monotone mappings on complete dioids any constraint can be expressed as an implicit inequality involving the state vector. We also prove the convergence of the computation. However, it must be noted that the obtained controllers are not necessarily minimal. We apply this method to the timetable synthesis of transportation systems. Future works should refine the control method as well as the model for the case of uncontrollable transitions and non linear constraints.

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