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Application of Lie groups in the study of beam and plate vibrations

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Abstract: Lie groups are continuous groups that have a structure of differential manifold. At the community of mechanical engineers are known their applications in numerical treatment of finite rotations. They are also used for solution of differential equations and for study of their properties. It is possible to use them for creation of new solutions from the solutions that are known. In the paper are given some basic properties of equations describing vibrations of beams and plates from this point of view.

Keywords: Lie group, Symmetry analysis, Differential equation, Vibration, Beam, Plate

1. Introduction

The theory of continuous group is known from the end of 19. century. These groups are named according to Sophus Lie (1842-1899) and they are now known as Lie groups. Continuous group theory and corresponding Lie algebras play an important role in understanding of the structure of differential equations and in finding their exact analytic solutions [1,5,8]. Lie groups are applied in various areas of mathematics, physics, and mechanics [2,9-15]. In the following we shortly describe the main relations of this theory. Details of the theory can the reader find in [1,5,8].

2. Lie groups and differential equations

A Lie group is a manifold equipped with a structure of a group. For any two points *a* and *b* in the manifold, there exists a multiplication *ab* and this group operation is consistent with the continuous structure of the manifold.

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Next, we will use this theory for studying transformations of solutions of partial differential equations. The finite transformations of a group with parameter $\varepsilon \in R$ can be written in the forms

$$\begin{aligned} \overline{x}_i &= \varphi_i(\mathbf{x}, \mathbf{u}, \varepsilon), \\ \overline{u}^\alpha &= \psi^\alpha(\mathbf{x}, \mathbf{u}, \varepsilon), \end{aligned} \tag{1}$$

where $\mathbf{u} = (u^1, ..., u^m)$ are functions which depend on independent variables $\mathbf{x} = (x_1, ..., x_n)$. The system of partial differential equations we define by

$$\Omega_v\left(x_i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, ..., u_{i_1,...,i_k}^\alpha\right) = 0, \tag{2}$$

$v = 1, ..., N$, where

$$u_{i_1,...,i_n}^\alpha = \frac{\partial^{i_1+...+i_n} u^\alpha}{\partial x_1^{i_1} ... \partial x_n^{i_n}} \tag{3}$$

are partial derivates. In order to formulate the conditions for the invariance of equations (2), the infinitesimal generator (or Lie point symmetry vector field)

$$\mathbf{U} = \xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{4}$$

is defined. Here and also in all paper the summation over twice-occurring indices is assumed. The infinitesimal quantities in (4) are defined by

$$\xi_i = \left.\frac{\partial \varphi_i(\mathbf{x}, \mathbf{u}, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \eta^\alpha = \left.\frac{\partial \psi^\alpha(\mathbf{x}, \mathbf{u}, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{5}$$

If we expand (1) around the value $\varepsilon = 0$ which corresponds to the group identity, then we have relations

$$\begin{aligned} \overline{x}_i &= x_i + \varepsilon \xi_i(\mathbf{x}, \mathbf{u}) + o(\varepsilon^2) \\ \overline{u}^\alpha &= u^\alpha + \varepsilon \eta^\alpha(\mathbf{x}, \mathbf{u}) + o(\varepsilon^2). \end{aligned} \tag{6}$$

The *k*-th prolongation of a vector field $\mathbf{U}$ is defined as

$$\mathbf{U}^{(k)} = \mathbf{U} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + ... + \zeta_{i_1...i_k}^\alpha \frac{\partial}{\partial u_{i_1...i_k}^\alpha}, \tag{7}$$

where the functions $\zeta_{i_1...i_k}^\alpha$ describe the transformations of partial derivatives of order *k*. For these functions we have the relations

$$\zeta_i^\alpha = \mathrm{D}_i(\eta^\alpha) - u_s^\alpha \mathrm{D}_i(\xi_s) \tag{8}$$

and

$$\zeta^{\alpha}_{i_1 \dots i_k} = \mathrm{D}_{i_k}\left(\zeta^{\alpha}_{i_1 \dots i_{k-1}}\right) - u^{\alpha}_{i_1 \dots i_{k-1}},_s \mathrm{D}_{i_k}\left(\xi_s\right), \tag{9}$$

where

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ki}^{\alpha}\frac{\partial}{\partial u_k^{\alpha}} + u_{kli}^{\alpha}\frac{\partial}{\partial u_{kl}^{\alpha}} \dots \tag{10}$$

is the operator of total differentiation with respect to x_i. The system of differential equations (2) is invariant under the transformations of a one-parameter group with the infinitesimal generator $\mathbf{U}$ if ξ_i and η^{α} are determined from the constraints

$$\mathbf{U}^{(k)}\Omega_v = 0, \tag{11}$$

where all $\Omega_v = 0$.

The action of Lie group on variables $\mathbf{x}, \mathbf{u}$ is given by relation

$$(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = g_{\varepsilon}.(\mathbf{x}, \mathbf{u}) = (\boldsymbol{\varphi}_{\varepsilon}(\mathbf{x}, \mathbf{u}), \boldsymbol{\psi}_{\varepsilon}(\mathbf{x}, \mathbf{u})) = e^{\varepsilon \mathbf{U}}(\mathbf{x}, \mathbf{u}) \tag{12}$$

Greek letter ε does not symbolize partial differentiation according to ε. This letter is in this paper reserved for the parameter of a group.

For a parameter ε the transformation of the function $\mathbf{f}$ by group element g_{ε} is then given by

$$\bar{\mathbf{u}} = \bar{\mathbf{f}}_{\varepsilon}(\bar{\mathbf{x}}) = (g_{\varepsilon}\mathbf{f})(\bar{\mathbf{x}}) = [\boldsymbol{\psi}_{\varepsilon} \circ (\mathbf{1} \times \mathbf{f})] \circ [\boldsymbol{\varphi}_{\varepsilon} \circ (\mathbf{1} \times \mathbf{f})]^{-1}(\bar{\mathbf{x}}). \tag{13}$$

Here $\mathbf{1}$ is the identity function $\mathbf{1}(\mathbf{x}) = \mathbf{x}$.

Calculation of Lie vector fields is tedious work. It involves a large amount of symbolic calculation that is better done by computer. For this purpose the computer program LIE [3,4] has been used.

3. Beam

The transverse vibrations of a beam is described by equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{A\rho}{EI}\frac{\partial^2 u}{\partial t^2} = 0, \tag{14}$$

where E is the modulus of elasticity, I is the moment of inertia of a cross section, A is the cross sectional area, and ρ is the mass per unit length. It is assumed that vibrations are small and perpendicular to the x axis. The transverse deflection at any point x on the length of the beam and at any time t is represented by a function $u(x,t)$.

For equation (14) the program LIE gives us the following Lie symmetry vector fields

$$\begin{aligned}
\mathbf{U}_1 &= \alpha(x,t)\frac{\partial}{\partial u}, \\
\mathbf{U}_2 &= \frac{\partial}{\partial t}, \\
\mathbf{U}_3 &= \frac{\partial}{\partial x}, \\
\mathbf{U}_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\
\mathbf{U}_5 &= u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},
\end{aligned} \tag{15}$$

where $\alpha(x,t)$ is any solution of equation (14). Now we will create according to equation (12) the transformation groups that belong to Lie vectors (15).

Let us take as an example generator

$$\mathbf{U}_5 = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \ . \tag{16}$$

At first we will search how the group generated by Lie vector $\mathbf{U}_5$ transforms the independent variable *x*. For this we use relations

$$\begin{aligned}
&\mathbf{U}_5(x) = x \\
&\mathbf{U}_5^2(x) = \mathbf{U}_5(x) = x \\
&\qquad\vdots \\
&\mathbf{U}_5^n(x) = x \ ,
\end{aligned} \tag{17}$$

which together with equation (12) give us

$$\bar{x} = x + \frac{\varepsilon}{1!}x + \frac{\varepsilon^2}{2!}x + \frac{\varepsilon^3}{3!}x + \ldots = x\left(1 + \frac{\varepsilon}{1!} + \frac{\varepsilon^2}{2!} + \frac{\varepsilon^3}{3!} + \ldots\right) = xe^{\varepsilon}. \tag{18}$$

For transformation of independent variable *t* we have relations

$$\begin{aligned}
&\mathbf{U}_5(t) = 2t \\
&\mathbf{U}_5^2(t) = \mathbf{U}_5(2t) = 4t \\
&\mathbf{U}_5^3(t) = \mathbf{U}_5(4t) = 8t \\
&\qquad\vdots
\end{aligned} \tag{19}$$

which result in

$$\bar{t} = t + \frac{\varepsilon}{1!}2t + \frac{\varepsilon^2}{2!}4t + \frac{\varepsilon^3}{3!}8t + \ldots = t\left(1 + \frac{(2\varepsilon)}{1!} + \frac{(2\varepsilon)^2}{2!} + \ldots\right) = te^{2\varepsilon}. \tag{20}$$

Similarly for dependent variable u we have

$$\bar{u} = ue^{\varepsilon}\ . \tag{21}$$

The one-parameter groups that belong to all Lie vectors (15) are

$$\begin{aligned}
&G_1 : (x,t,u) \mapsto (x,t,u+\varepsilon\,\alpha(x,t))\\
&G_2 : (x,t,u) \mapsto (x,t+\varepsilon,u)\\
&G_3 : (x,t,u) \mapsto (x+\varepsilon,t,u)\\
&G_4 : (x,t,u) \mapsto (e^{\varepsilon}x, e^{2\varepsilon}t, u)\\
&G_5 : (x,t,u) \mapsto (e^{\varepsilon}x, e^{2\varepsilon}t, e^{\varepsilon}u)
\end{aligned} \tag{22}$$

Because every group from (22) is symmetry group, from equation (13) follows, that if $f(x,t)$ is the solution of equation (14), then functions

$$\begin{aligned}
&u^{(1)} = f(x,t) + \varepsilon\,\alpha(x,t)\\
&u^{(2)} = f(x,t-\varepsilon)\\
&u^{(3)} = f(x-\varepsilon,t)\\
&u^{(4)} = f(e^{-\varepsilon}x, e^{-2\varepsilon}t)\\
&u^{(5)} = e^{\varepsilon} f(e^{-\varepsilon}x, e^{-2\varepsilon}t)
\end{aligned} \tag{23}$$

are solutions, too.

4. Plate

Free vibration of a quadrilateral plate is described by equation

$$\nabla^2\nabla^2 u + \frac{\rho\, h}{D}\frac{\partial^2 u}{\partial t^2} = 0, \tag{24}$$

where ρ is the mass per unit volume, h is the plate thickness, and

$$D = E\,h^3 / \{12(1-v^2)\}\cdot$$

Here, E is Young modulus and v is Poisson ratio. For the plate described in coordinates x, y we have from the program LIE the following generators

$$
\begin{aligned}
&\mathbf{U}_1 = \beta(x,y,t)\frac{\partial}{\partial u},\\
&\mathbf{U}_2 = \frac{\partial}{\partial y},\\
&\mathbf{U}_3 = \frac{\partial}{\partial x},\\
&\mathbf{U}_4 = \frac{\partial}{\partial t},\\
&\mathbf{U}_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\\
&\mathbf{U}_5 = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t},\\
&\mathbf{U}_7 = u\frac{\partial}{\partial u}.
\end{aligned} \tag{25}
$$

Here $\beta(x,y,t)$ is any solution of equation (24).

Let us compute less trivial example of transformations that belong to Lie vector

$$\mathbf{U}_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{26}$$

For transformation of independent variable x we have

$$
\begin{aligned}
&\mathbf{U}_5(x) = -y\\
&\mathbf{U}_5^2(x) = -\mathbf{U}_5(y) = -x\\
&\mathbf{U}_5^3(x) = -\mathbf{U}_5(x) = y\\
&\mathbf{U}_5^4(x) = \mathbf{U}_5(y) = x\\
&\quad\vdots
\end{aligned} \tag{27}
$$

$$\bar{x} = x + \varepsilon(-y) + \frac{\varepsilon^2}{2!}(-x) + \frac{\varepsilon^3}{3!}(y) + \frac{\varepsilon^4}{4!}(x) + \ldots = x\cos\varepsilon - y\sin\varepsilon. \tag{28}$$

For independent variable y we have

$$
\begin{aligned}
&\mathbf{U}_5(y) = x\\
&\mathbf{U}_5^2(y) = -y\\
&\mathbf{U}_5^3(y) = -x\\
&\mathbf{U}_5^4(y) = y\\
&\quad\vdots
\end{aligned} \tag{29}
$$

$$\bar{y} = y + \varepsilon(x) + \frac{\varepsilon^2}{2!}(-y) + \frac{\varepsilon^3}{3!}(-x) + \frac{\varepsilon^4}{4!}(y) + \ldots = y\cos\varepsilon + x\sin\varepsilon . \tag{30}$$

All transformation groups that belong to vectors (25) are

$$\begin{aligned}
&G_1 : (x, y, t, u) \to (x, y, t, u + \varepsilon\beta(x, y, t)) \\
&G_2 : (x, y, t, u) \to (x, y + \varepsilon, t, u) \\
&G_3 : (x, y, t, u) \to (x + \varepsilon, y, t, u) \\
&G_4 : (x, y, t, u) \to (x, y, t + \varepsilon, u) \\
&G_5 : (x, y, t, u) \to (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, t, u) \\
&G_6 : (x, y, t, u) \to \left(e^{\varepsilon}x, e^{\varepsilon}y, e^{2\varepsilon}t, e^{\varepsilon}u\right) \\
&G_7 : (x, y, t, u) \to \left(x, y, t, e^{\varepsilon}u\right)
\end{aligned} \tag{31}$$

and any solution $f(x, y, t)$ of the equation (24) can be transformed into solutions

$$\begin{aligned}
&u^{(1)} = f(x, y, t) + \varepsilon\beta(x, y, t) \\
&u^{(2)} = f(x, y - \varepsilon, t) \\
&u^{(3)} = f(x - \varepsilon, y, t) \\
&u^{(4)} = f(x, y, t - \varepsilon) \\
&u^{(5)} = f(x\cos\varepsilon + y\sin\varepsilon, -x\sin\varepsilon + y\cos\varepsilon, t, u) \\
&u^{(6)} = e^{\varepsilon} f\left(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-2\varepsilon}t\right) \\
&u^{(7)} = e^{\varepsilon} f(x, y, t).
\end{aligned} \tag{32}$$

5. Conclusion

In the paper were analyzed differential equations that describe free vibrations of beams and plates. The analysis was accomplished by application of Lie group theory. For the given differential equations the Lie vectors and their nonvanishing generators were computed by computer program LIE. From the Lie vectors the transformation groups were computed. Lie vectors can also be used for constructing special solutions that are obtained from the symmetry of differential equations. These are known as similarity solutions. However, this aspect of Lie theory was not observed in this paper.

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