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Pure Shear Experiment for Rubberlike Materials

Bohdana Marvalova,[1] Hoang Sy Tuan,[1] Iva Petrikova[1]

Abstract: Mechanical engineers are familiar with the simple shear – state of stress induced e.g. by twisting of a thin tube. On the other hand, the concept of the pure shear used by experimentalists dealing with rubber and other elastomers is esoteric a little bit. The pure shear experiment performed is not what most of us would expect. It appears at first glance to be nothing more than a tensile test with a very wide specimen [1]. Our paper tries to shed some light upon this concept. Kinematics of simple shear and pure shear is compared from the viewpoint of theory of finite deformations. The role of the pure shear experiment in testing of rubber-like solids is explained. Results of pure shear experiment are presented.

Keywords: Experimental, Pure Shear, Elastomers

1. Introduction

When we look for papers concerning pure shear and simple shear we usually fall in fervent discussions amongst geologists about the McKenzie's pure shear model [2] and the Wernicke's simple shear model [3] of the deformation process that affected the continental lithosphere in the geologic past. Both models assume a constant volume deformation. In the case of pure shear the material uniformly elongates in one direction and uniformly shortens in one perpendicular direction while its size does not change in the second perpendicular direction. In the simple shear, on the other hand, the material moves parallel to a given direction like a deck of cards and the size perpendicular to the shear plane does not shorten.

There is a great variety of shear test methods developed and used in experimental mechanics and all experimentalists know how difficult it is to obtain a reasonably pure and uniform shear stress state in the test specimen. If we are dealing with materials undergoing small strains, the shear test is usually seen in terms of state of stress and kinematics plays a minor role. On the other hand, in the case of rubber or elastomeric materials which are capable of large deformations and are considered as incompressible, it is appropriate to examine the shear test from the kinematical view-point.

Experimental measurements of the response of rubber-like materials are conducted usually under plane stress state.

[1] *Dept. of Applied Mechanics, Technical University of Liberec; Studentska 2, 46117 Liberec, Czech Republic; bohda.marvalova@tul.cz, situanh@yahoo.com, iva.petrikova@tul.cz*

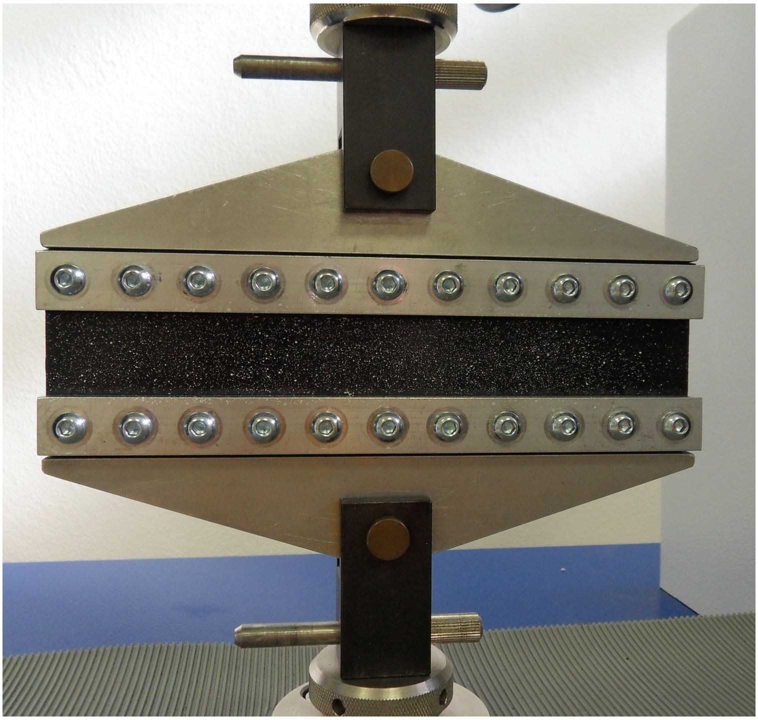

Fig. 1. Pure shear test device.

Rivlin and Saunders [4] described a tensile test with a specimen cut from a sheet of vulcanised rubber that was much wider than long similar to Fig. 1. They denominate as pure shear this homogeneous deformation in which one of the stretches in the plane of the sheet is maintained at unity, while the other is varied. Their specimen had dimensions 95x20x0.87 mm. They reported that although the rectangular form of specimen was not accurately preserved, the width of the specimen at the horizontal axis varied only by 3% while the stretch in the direction of tension reached value λ=2.2.

The state of stress in the pure shear experiment is biaxial – however both principal stresses are tensile in contrast to the simple shear caused by torsion in a thin tube.

The paper is structured as follows. In Section 2 the basic kinematics of the pure shear and simple shear is explained. In Section 3 the constitutive relations useful in evaluation of pure shear experiment are presented. Section 4 presents results of pure shear experiments. Finally, some concluding remarks are addressed in Section 5.

2. Kinematics

Pure shear kinematics is illustrated in Fig. 2a. We suppose that the material submitted to pure shear is incompressible then the product of principal stretches $\lambda_1\lambda_2\lambda_3 = 1$. If the principal stretch in the direction of tension is $\lambda_2 = \lambda$ and the stretch in the direction of width of specimen is $\lambda_1 = 1$ (the width does not change) then the stretch in the direction of thickness is $\lambda_3 = \lambda^{-1}$. We describe such motion by relations $x_1 = X_1$, $x_2 = \lambda X_2$, $x_3 = \lambda^{-1} X_3$. The deformation gradient ***F***, the right stretch tensor ***U*** and the rotation tensor ***R*** of pure shear are:

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}, \quad \boldsymbol{U} = \boldsymbol{F}, \quad \boldsymbol{R} = \boldsymbol{I}, \tag{1}$$

where ***I*** is the unit tensor. We see that the principal axes do not rotate in course of the pure shear experiment. It is not so in the case of simple shear as we will see thereinafter.

The simple shear motion is depicted in Fig. 2b. We suppose that the dimension of specimen in the direction of third axis does not change. The motion is described by relations $x_1 = X_1 + k X_2$, $x_2 = X_2$, $x_3 = X_3$, where $k = \tan(\gamma)$. The deformation gradient ***F*** and the right Cauchy-Green deformation tensor ***C*** are

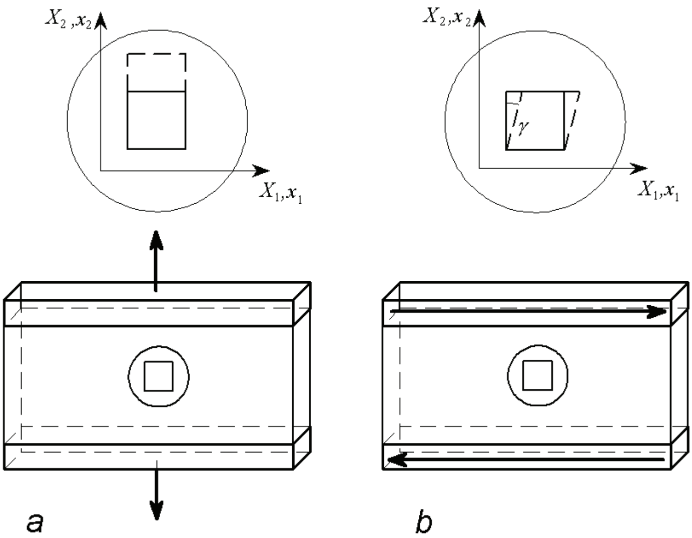

Fig. 2. Kinematics of pure shear (a) and simple shear (b).

$$\boldsymbol{F}=\begin{bmatrix}1 & k & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \boldsymbol{C}=\boldsymbol{F}^{\boldsymbol{T}}\boldsymbol{F}=\begin{bmatrix}1 & k & 0\\ k & 1+k^2 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{2}$$

We see that $det\boldsymbol{F}$ = 1 then the motion in simple shear is isochoric. We shall determine principal stretches so the components of the right stretch tensor $\boldsymbol{U}$ in the principal axes. We perform first the spectral decomposition of the deformation tensor $\boldsymbol{C}=diag(\lambda_1^2,\lambda_2^2,\lambda_3^2)$. We know that $\lambda_3=1$ (plane deformation) and the remaining eigenvalues fulfil equation

$$\begin{vmatrix}1-\lambda^2 & k\\ k & \left(1+k^2\right)-\lambda^2\end{vmatrix}=0 \quad\Rightarrow\quad \lambda^4-\lambda^2\left(2+k^2\right)+1=0$$

$$\Rightarrow\quad \lambda^2+\lambda^{-2}-\left(2+k^2\right)=0 \quad\Rightarrow\quad \left(\lambda+\lambda^{-1}\right)^2=k^2 \tag{3}$$

$$\Rightarrow\quad \lambda_2=\lambda_1^{-1}.$$

It is evident that the stretches in the principal directions in simple shear have the same pattern as in pure shear. i.e.

$$U=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda^{-1} & 0\\ 0 & 0 & 1\end{bmatrix},\ \text{where } \lambda=\left(\pm k+\sqrt{k^2+4}\right)/2. \tag{4}$$

The deformation in simple shear is a combination of a pure stretch and a pure rotation. For plane deformations, the components of rotation tensor can be directly computed [5] by

$$\boldsymbol{R}=\frac{\begin{bmatrix}F_{11}+F_{22} & F_{12}-F_{21}\\ F_{21}-F_{12} & F_{22}+F_{11}\end{bmatrix}}{\sqrt{\left(F_{11}+F_{22}\right)^2+\left(F_{12}-F_{21}\right)^2}}=\frac{1}{\sqrt{4+k^2}}\begin{bmatrix}2 & k\\ -k & 2\end{bmatrix}$$

$$\Rightarrow \cos\left(\theta\right)=\frac{2}{\sqrt{4+k^2}},\ \sin\left(\theta\right)=\frac{-k}{\sqrt{4+k^2}} \tag{5}$$

$$\Rightarrow \tan\left(\theta\right)=\frac{-k}{2}=-\frac{1}{2}\tan\left(\gamma\right).$$

Then for small shear deformation γ the angle of polar rotation $\theta=-\gamma/2$. Note that a positive γ corresponds to a negative (clockwise) polar rotation angle.

3. Constitutive relations

The stress response of hyperelastic materials is derived from a strain-energy function $\Psi(\boldsymbol{F})$ of the deformation gradient. A variety of suitable strain-energy functions can be found in literature [6, 7]. We confine us to well-known Ogden model [8] for incompressible rubberlike materials

$$\Psi(\lambda_1,\lambda_2,\lambda_3)=\sum_{n=1}^{N}\frac{\mu_n}{\alpha_n}\left(\lambda_1^{\alpha_n}+\lambda_2^{\alpha_n}+\lambda_3^{\alpha_n}-3\right),$$
$$2\mu=\sum_{n=1}^{N}\mu_n\alpha_n, \tag{6}$$

where μ denotes the shear modulus. Material parameters must satisfy the following essential conditions

$$\mu_n\alpha_n>0. \tag{7}$$

Components of Cauchy stress for an incompressible material are derived [6] from the strain energy function $\Psi(\lambda_1,\lambda_2,\lambda_3)$

$$\sigma_a=-p+\lambda_a\frac{\partial\Psi}{\partial\lambda_a},\quad a=1,2,3, \tag{8}$$

where p is unknown hydrostatic pressure which is determined from equilibrium conditions at the boundary. In the case of pure shear the faces of specimen are without any loading $\sigma_3=0$ thus

$$p=\sum_{n=1}^{N}\mu_n\lambda_3^{\alpha_n}. \tag{9}$$

If we substitute $\lambda_1=\lambda,\ \lambda_2=1,\ \lambda_3=\lambda^{-1}$ and p into Eqs. (8) we obtain stress-strain relations

$$\sigma_1=\sum_{n=1}^{N}\mu_n\left(\lambda^{\alpha_n}-\lambda^{-\alpha_n}\right),$$
$$\sigma_2=\sum_{n=1}^{N}\mu_n\left(1-\lambda^{-\alpha_n}\right). \tag{10}$$

The material parameters μ_n (shear moduli) and α_n (dimensionless constants) have to be determined from experimental stress-deformation data. There is usually no need for more than three pairs of constants in practice to achieve a good correlation with the experimental data.

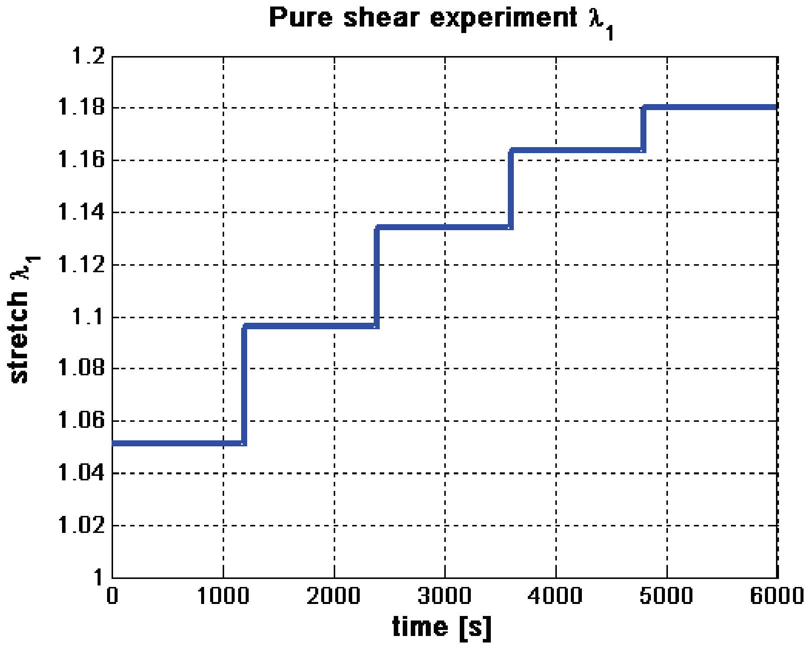

Fig. 3. Strain controlled loading.

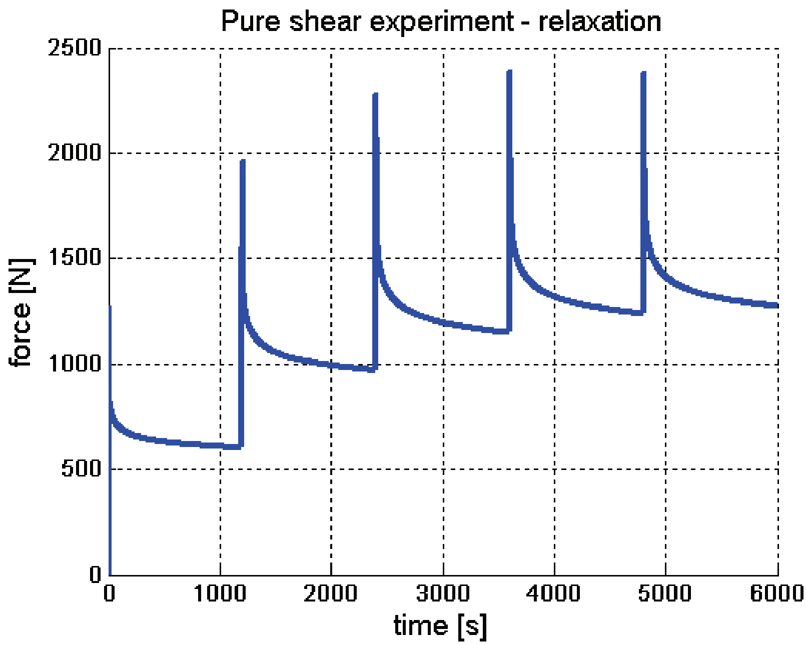

Fig. 4. Force response.

4. Experiment

Pure shear experiment [9] depicted on Fig. 1 was performed on the testing machine TIRA 2810. The rectangular specimens 220x20x2.9 mm were cut from a sheet of styrene-butadiene rubber SBR/E of hardness 74 Shore A.

The loading was strain controlled and specimens were loaded gradually with 20 min relaxation delay between every loading step see Fig. 3. The force F response is shown on Fig. 4. The values of stress at the end of relaxation periods are regarded as the equilibrium stresses Fig.5. The stresses were evaluated considering the change of cross-sectional area A_0

$$\sigma_1 = \frac{F\lambda^2}{A_0}. \tag{11}$$

The parameters of Ogden model were fitted from experimental stress-deformation data. The set of overdetermined equations Eqs. (10_1) was solved by means of Matlab Optimization Toolbox. The equations are linear in parameters μ_n and nonlinear in parameters α_n thus a succession of approximate solutions has been generated by linear and nonlinear last squares alternately till conditions of optimization were satisfied. We remind that the requirements in Eqs. (7) are crucial and decisive for the admissibility of every approximate solution. The comparison of the experimental and fitted stress σ_1 is on Fig. 6 together with the stress σ_2 calculated from Eqs. (10_2).

5. Conclusion

Kinematics of pure shear and simple shear were compared. Constitutive relations of pure shear based on Ogden strain energy function were reminded. The results of the pure shear experiment were presented. Pure shear is a simple and easy experimental arrangement producing biaxial state of stress.

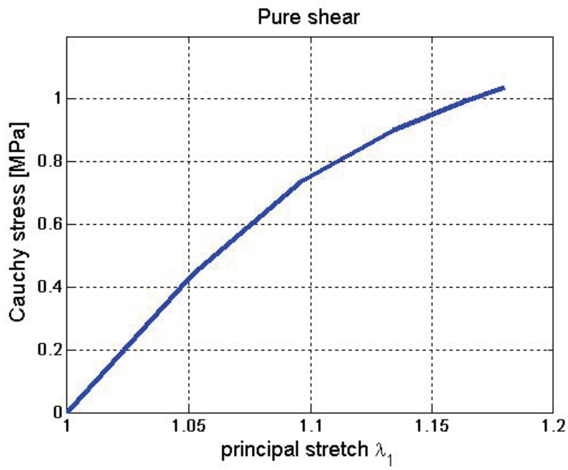

Fig. 5. Stresses at the end of relaxation period.

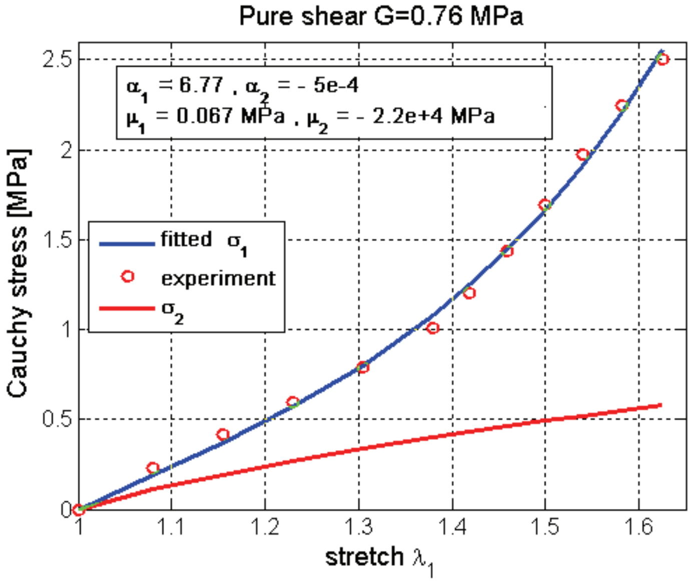

Fig. 6. Comparison of experimental and fitted values of the loading stress σ_1 and the graph of the stress in transversal direction σ_2 calculated from Eqs. (10_2).

Acknowledgements

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