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Dimensional Analysis and Modelling in Mechanics of Solids

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Abstract: The article deals with model design conditions. The theory of models is developed using findings of dimensional analysis. It is shown that any physical problem can be described by a set of dimensionless parameters. Model design conditions are presented.

Keywords: Dimensional analysis, Dimensional matrix, Dimensionless set of π terms, Similitude, Modelling laws

1. Introduction

Modelling of engineering problems can be very often a way to solve them. An engineering model represents a sure system giving possibility to predict the behaviour of the examined engineering problem. In comparison with mathematical models physical ones give possibility made analysis on the real structures or on the corresponding length scaled models. They provides tool to eliminate many technical difficulties in the case of mathematical modelling. The concept of similitude provides tool when results obtained on the model can be accepted to describe the behaviour of the corresponding structure or prototype. Necessary relationships are called similarity requirements and needed principles will be developed through the dimensional analysis and dimensionless π terms.

2. Dimensional analysis

Let us consider a physical problem, involving variables x_i, $i = 1, 2, 3, ..., n$ expressing physical quantities as force, mass, time atc., to be solved. Then that problem can be described by some function [1] among the variables

$$y = y(x_i) \ . \tag{1}$$

Variables are in that case generally taking into account all quantities influencing a solved problem, regardless if they are really variable or constant as for example Poisson ratio, Young´s modulus of elasticity. The form of an equation describing a physical or engineering problem respectively is independent of the chosen units of quantities involved. Such equations are called complete and they are dimensional

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homogeneous. It means if the function Eq. (1) is expressed in a form of series, then all terms involved there are of the same dimension.

By the Nusselt´s rule Eq. (1) can be expressed as

$$y = x_1^{e1} \;\; x_2^{e2} \ldots x_n^{e_n} \tag{2}$$

where e_i, $i = 1, 2, 3, ..., n$ are up to now unknown exponents.

Let us assume that among quantities x_i describing the solving problem is *k* so called primary quantities with primary units $\lfloor L_j \rfloor$, $j = 1, 2, ..., k$. These primary quantities are then used to provide a qualitative description of so called secondary quantities. In the case of Mechanics of Solids the primary quantities are: mass *m*, length *l* and time *t*. A dimension of an arbitrary secondary quantity x_i can be expressed as a formal product of primary dimensions with the exponent resulting from the definition of taking into account quantity. Then the dimension of the x_i quantity can be expressed as.

$$[X_i] = [L_1]^{\alpha_i} \; [L_2]^{\beta_i} \; [L_3]^{\gamma_i} \ldots\ldots [L_k]^{\varphi_i} \tag{3}$$

And the dimension of the quantity y, Eqn. (1), can be expressed as

$$\begin{aligned} [Y] = & \left([L_1]^{\alpha_1} \; [L_2]^{\beta_1} \ldots [L_j]^{\delta_1} \ldots. [L_k]^{\varphi_1}\right)^{e_1} \; \left([L_1]^{\alpha_2} \; [L_2]^{\beta_2} \ldots [L_j]^{\delta_2} \ldots. [L_k]^{\varphi_2}\right)^{e_2} . \\ & \ldots\ldots\ldots \left([L_1]^{\alpha_n} \; [L_2]^{\beta_n} \ldots.. [L_j]^{\delta_n} \ldots\ldots [L_k]^{\varphi_n}\right)^{e_n} \end{aligned} \tag{4}$$

Eq. (4) can be rewritten as

$$[Y] = [L_1]^{\sum_1^n \alpha_i e_i} \; [L_2]^{\sum_1^n \beta_i e_i} \ldots.. [L_j]^{\sum_1^n \delta_i e_i} \ldots.. [L_k]^{\sum_1^n \varphi_i e_i} \; . \tag{5}$$

To obtain dimensionless terms in Eq. (5), the total exponent of each of term there must vanish, i.e.

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i \; e_i = 0 \, , \\ &\sum_{1}^{n} \beta_i \; e_i = 0 \;\; , \\ &\qquad \vdots \\ &\sum_{1}^{n} \varphi_i \; e_i = 0 \;\; . \end{aligned} \tag{6}$$

Then the Eq. (2) is dimensionless. There are *n* unknown exponents e_i involved in Eq. (6) and there are only *k* of equations. Therefore one has to assign $n-k$ quite arbitrary values to unknown exponents, to be able to determine values of remaining other k ones using Eq. (6). Arbitrary exponents are chosen as small positive or negative integer or zero.

If in the manner mentioned above determined exponents are inserted into Eq. (2) one obtains dimensionless parameters.

$$\pi = x_1^{e_1} \; x_2^{e_2} \; x_3^{e_3} \ldots x_n^{e_n} \,. \tag{7}$$

They were by Buckingham called „pi“ terms and he as the first used the symbol π for them.

The solved problem $y = y(x_i), \quad i = 1, 2, ..., n$ can then be reduced to a relationship of m non-dimensional parameters expressed as

$$\phi(\pi_j) = 0, \quad j = 1, 2, ..., m \,. \tag{8}$$

As it is obvious variables π_j are fewer in number than origin number n of variables x_i. By the Buckingham conclusion, then there are only

$$m = n - r \tag{9}$$

independent π terms. They can be used to solve taking into account problem. r is generally rank of so called dimensional matrix [**D**]. Eq. (6) can be then presented in matrix input as

$$[\mathbf{D}]\{\mathbf{e}\} = \{\mathbf{0}\} \;, \tag{10}$$

where $\{\mathbf{e}\} = [\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, ..., \mathbf{e}_n]^T$ is the vector of unknown exponents.

Usually the value of r in Eq. (9) is equal to the number k of dimensions of primary quantities involved in the n variables quantities, see later. Construction of the dimensional matrix for n variables x_i describing the problem and k primary units involved in solved problem is obvious from Eq. (11).

$$[\mathbf{D}] = \begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & \cdots & x_i & \cdots & x_n \end{array} \\ \hline \begin{array}{c} L_1 \\ L_2 \\ \vdots \\ \vdots \\ L_k \end{array} & \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_i & \cdots & \alpha_n \\ \beta_1 & \beta_2 & & \beta_i & & \beta_n \\ & & & & & \\ & & & & & \\ \varphi_1 & \varphi_2 & & \varphi_i & & \varphi_n \end{bmatrix} \end{array} \,, \tag{11}$$

where $\alpha_i, \beta_i, ..., \varphi_i, \; i = 1, 2, ..., n$ are exponents in Eq. (3) resulting from definitions of corresponding involved quantities. As it was mentioned above usually is the rank of dimensional matrix equal to the number k of dimensions of primary quantities involved in n variables quantities describing the solved problem. Generally the rank of a matrix is equal to the highest matrix determinant order different from zero. There exist rule for its determination. But in the case of physical problem the way is very simple. To achieve the rule $r = k$ is very simple. It is sufficient to order the primary quantities as variables in matrix columns in the same order as to them corresponding units in dimensional matrix rows.

In the case of Mechanics of Solids there exist two approaches:

- Dynamic approach with primary units: mass $m[kg]$, length $l[m]$ and time $t[s]$. If the order of variables $x_1 = m,\ x_2 = l,\ x_3 = t$ in the first three columns is the same as order of their units in rows, it is obvious that $r = k$ and the rank of the dimensional matrix [D] is $r = 3$ and it is equal to the number of involved in the dynamic problems primary units.
- Static approach, when problems depend on force $F[N] = [kg\, m\, s^{-2}]$ only and length $l[m]$. There, in this case, are not as separate quantities mass *m*, and time *t*. They are only included in the secondary unit [*N*]. Therefore from the point of view of dimensional analysis, unit of force [*N*] can be taking into account as a basic unit and it is obvious that the rank of the matrix [**D**] is then $r = 2$ in this case.

It is obvious, that there is not problem to create π terms directly. But always it is necessary to use only the number of independent numeric *m*. It is recommended that if there are *k* primary dimensions of involved variables a maximum of only $k+1$ quantities should be included in each π term [3].

It is necessary to create one group of π terms in such way to contain only one of variables of our interest and second group containing variables describing the inspected system.

2.1. Example

As an example let us consider the cantilever beam with the hollow rectangular cross section subjected to concentrated loads, (Fig. 1).

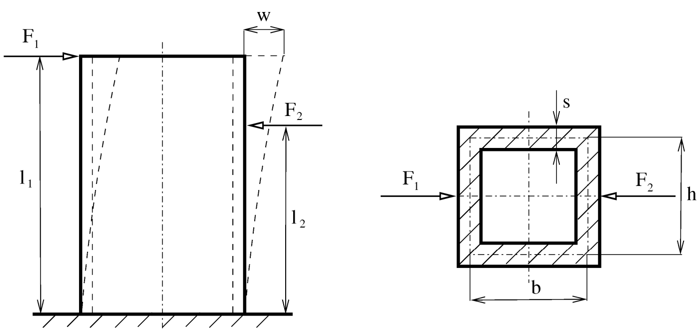

Fig. 1. Cantilever with the hollow rectangular cross section.

The general problem in elastic field is necessary consider as two problems, because the deformations are dependent upon Young´s modulus of elasticity, the stress distribution in a body is independent of it. Then the deformation of the beam w will be a function of geometry of the body, loads and modulus of elasticity E

$$w = w(l_1, l_2, h, b, s, F_1, F_2, E) \ , \tag{12}$$

in our case. On the other hand the stress components would be functions of geometry of the body and loads only. Let us take into account, that the deflection w (Fig. 1) is the quantity of our interest. Then there are 9 variables in our case and there will be $m = n - r = 7$ independent π terms. Let us choose variable of our interest w to express π_1

$$\pi_1 = \frac{w}{l_1} \ . \tag{13}$$

Then π terms determining the geometry shape of the body and positions of acting forces can be created immediately as functions of chosen characteristic length l_1 and force F_1 for example, then

$$\pi_2 = \frac{l_2}{l_1}, \quad \pi_3 = \frac{h}{l_1}, \quad \pi_4 = \frac{b}{l_1}, \quad \pi_5 = \frac{S}{l_1} \ . \tag{14}$$

Similarly for acting forces

$$\pi_6 = \frac{F_2}{F_1} \ . \tag{15}$$

The remaining term π_7 then has to involve chosen acting force F_1, chosen length l_1 and material property expressed by modulus of elasticity E. Then π_7 will be

$$\pi_7 = l_1^{e1} \cdot F_1^{e2} \cdot E^{e3} \ . \tag{16}$$

Where exponents $\{e\} = [e_1, e_2, e_3]^T$ has to fulfil equation

$$[\mathbf{D}]\{\mathbf{e}\} = \{\mathbf{0}\} \ . \tag{17}$$

The dimensional matrix in this case is

$$[\mathbf{D}] = \begin{array}{c|c} & x_1 = l_1 \quad x_2 = F_1 \quad x_3 = E \\ \hline \begin{array}{c} L_1\,[N] \\ L_2\,[m] \end{array} & \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ \hline 1 & 0 & -2 \end{array} \right] \end{array} \tag{18}$$

There are only two equations available for determination of three unknown exponents. Then value of one of them has to be chosen. Deflection of a beam is a linear function of load, therefore let us chose $e_2 = 1$. Then Eq. (17) can be rewritten as

$$e_2 + e_3 = 0, \qquad e_1 - 2e_3 = 0 \ . \tag{19}$$

Inserting $e_2 = 1$ one obtains $e_3 = -e_2 = -1, \quad e_1 = 2e_3 = -2$.

Then inserting obtained exponents into Eq. (16) that equation will be

$$\pi_7 = \frac{F_1}{El_1^2} \ . \tag{20}$$

The defection *w* can be using π terms now expressed as

$$\pi_1 = \phi_1 (\pi_2, \pi_3, \pi_4, \pi_5 \ \pi_6, \pi_7) \tag{21}$$

or

$$\frac{w}{l} = \phi_1 \left(\frac{l_2}{l_1}, \frac{h}{l_1}, \frac{b}{l_1}, \frac{s}{l_1}, \frac{F_2}{F_1}, \frac{F_1}{El_1^2} \right) \ . \tag{21a}$$

3. Model similitude

The model similitude can be simply explained using the principles of dimensional analysis. It has been concluded that any problem of our interest can be solved using corresponding π terms set as

$$\phi(\pi_1, \pi_2, ..., \pi_m) = 0 \tag{22}$$

To solve that problem only knowledge of involved variables (depending on a character of the problem) is necessary. As it has resulted from dimensional analysis above, there are not necessary to know concrete values of involved variables to express π terms. Taking into account that fact, then two systems as a structure and the corresponding model can be described by a similar relationship as Eq. (21) and form of the function ϕ_1 will be same because of the same phenomenon solved in the case of the structure and the model.

If, for example, term π_1 was chosen as the variable that is to be determined using observation on the model, then the Eq. (21) can be rewritten for model as

$$\pi_{1M} = \phi_1 (\pi_{2M}, \pi_{3M}, ..., \pi_{mM}) \tag{23}$$

Then if the model (suffix *M*) and structure (suffix *S*) fulfil conditions

$$\pi_{2M} = \pi_{2S}, \ ...., \ \pi_{mM} = \pi_{mS}, \tag{24}$$

then Eq. (23) gives also

$$\pi_{1S} = \pi_{1M} \ . \tag{25}$$

The conditions

$$\pi_{iS} = \pi_{iM}, \quad i = 1, 2, .., m \tag{26}$$

express so called model design conditions. The necessary scale similarity of both model and structure results from the conditions of equality of corresponding π terms both of the structure and the model depending on their geometry.

For the taking into account beam as the example will be

$$\frac{w}{l_1} = \phi_1\left(\frac{l_2}{l_1}, \frac{h}{l_1}, \frac{b}{l_1}, \frac{s}{l_1}, \frac{F_2}{F_1}, \frac{F_1}{El_1^2}\right)$$

and the deflection of the real column in our case can be expressed as the function of the measured deflection on the model as

$$w_S = w_M \frac{l_{1S}}{l_{1M}},$$

when the corresponding π terms given by Eq. (14 and 15) fulfil in Eq. (24) are. The ratio of l_{1S}/l_{1M} expresses the length scale of model and structure and similarly for forces scale is F_{1S}/F_{1M} atc.

4. Conclusions

- Using dimensional analysis the solved problem can be described by a set of dimensionless π terms.
- Solution of the problem in dimensionless π terms leads to the simplification, when the number of involved variables is always reduced. Because of it corresponding experiments are performed more efficiently.
- From the explained above procedure it is obvious that the approach of dimensionless variables is general without any restriction on to a particular units system.
- The model design conditions can be simply expressed as $\pi_{iS} = \pi_{iM}$, $i = 1, 2, .., m$. It expresses the equality of corresponding π terms of the structure and the model.

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