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MODAL PARAMETER ESTIMATION FOR UNDERDAMPED LINEAR DYNAMIC SYSTEMS

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Abstract: Modal parameter estimation is the estimation of frequency, damping ratio, and modal coefficients from experimental data. Modal analysis techniques are a common method used to determine these properties. The Least-Squares Complex Exponential (LSCE) and the Eigensystem Realization Algorithm (ERA) are one of the popular methods of modal analysis techniques. This paper presents an experimental verification of the LSCE and ERA methods. The investigation focuses on the estimation of natural frequencies, damping ratio and modal coefficients. To investigate this, artificial analytical data were processed in MATLAB environment to estimate the modal parameters. The identified vibration parameters from the LSCE and ERA were compared with the values based on classical dynamic theory, and the natural frequency and damping ratio's percent of error were calculated.

Introduction

In many practical circumstances, the vibration characteristics of a dynamic system require to be understood and, subsequently, an accurate mathematical model needs to be derived. For a discrete linear dynamic system with lumped masses and mass-less elastic components, theory has been well developed to study the vibration characteristics. Apart from the approach of theoretical prediction to achieve an analytical model for the study of vibration characteristics of a dynamic system, another major approach is to establish an experimental model for the system by performing vibration tests and subsequent analysis on the measured data.

The present work discusses the estimation of modal parameters and modal parameter extraction technique. Two modal parameter extraction techniques were selected to perform this investigation; these are the LSCE and ERA.

Literature Review

The concept of discretization of an object and introduction of matrix analysis brought about a climax in theoretical modal analysis early in the last century [6]. The invention of the fast Fourier transform algorithm by *J. W. Cooly* and *J. W Tukey* in 1965 finally paved the way for rapid and prevalent application of the experimental technique in mechanical and structure dynamics [2], [6]. A vast amount of work along this line has been studied and published by many researchers. For example, *S. Fahey* and *J. Platt* in [4], [5] explain how to fit experimental data using SDOF and MDOF techniques. They concluded that SDOF techniques are attractive when performing quick field analysis or to provide initial estimate for more complex MDOF techniques. *J. N Jaung* and *R. Pappa* developed the ERA in 1985. The ERA is an extension of the *Ho-Kalman* algorithm that incorporates singular value decomposition to counteract inherent noise [4]. The general approaches to modal testing and their reviews can be found in [6] and reference [2] gives a broader explanation regarding modal parameter extraction techniques.

Artificial Simulated Data

The figure below shows a 3DOF theoretical model that was developed to investigate the effects of various eccentricities due to eccentric weight loading. This eccentric weight introduces rotational as well as vertical motions of the mechanical system. The equation of motions for the system is based upon a matrix differential equation (1) transformed into the domain of interest

$$\mathbf{M}\ddot{q}(t)+\mathbf{B}\dot{q}(t)+\mathbf{K}q(t)=F(t) \qquad \text{for j = 1, 2, 3.} \tag{1}$$

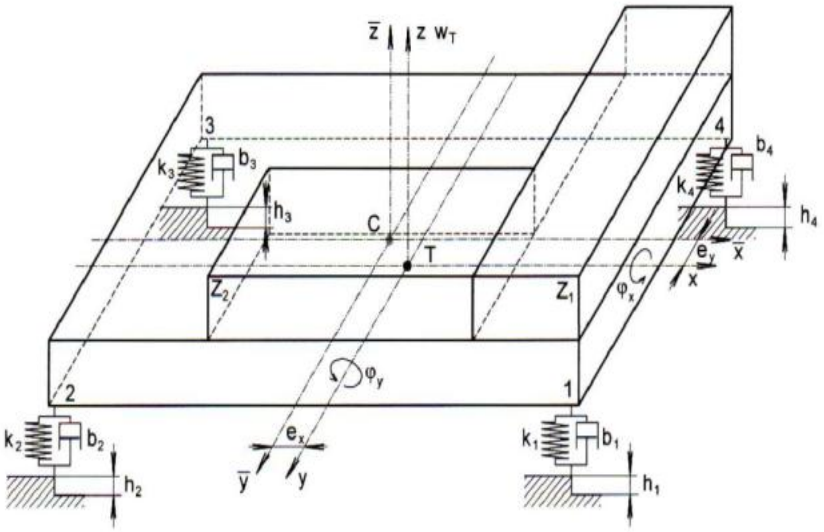

Figure. 1 – Analytical model

The general solution of the system of differential equations (1) can be obtain by Laplace integral transformation and after inverse transformation of the generalized coordinate, the form of the sum of convolution integral is obtained as follows

$$q_j(t)=\sum_{i=1}^{n}(-1)^{j+1}\sum_{k=1}^{n}\Bigg[K_{ji,k}\int_0^t F_i(\tau)e^{-\beta_k(t-\tau)}\cos\Omega_k(t-\tau)d\tau+$$

$$\frac{L_{ji,k}-\beta_k K_{ji,k}}{\Omega_k}\int_0^t F_i(\tau)e^{-\beta_k(t-\tau)}\sin\Omega_k(t-\tau)d\tau\ \Bigg] \tag{2}$$

The natural frequencies and mode shapes of a undamped system derived from equation (1) can be found by the solution of the classic eigenvalue - eigenvector problem [6] as

$$[\mathbf{K}-\omega_r^2\mathbf{M}]\phi_r=0 \qquad \text{for n = 1, 2, 3.} \tag{3}$$

Reference [1] and [2] gives a complete derivation of this method. Where **M** – is the mass matrix, **K** – is the stiffness matrix, **B** – damping matrix, $q=q_j^T=(w_c,\varphi_x,\varphi_y)^T$ – vector of the generalized coordinate, ω_n – is the natural frequencies and ϕ_r - mode shapes of the system. $F=(F_1(t),F_2(t),F_3(t))^T$ - is the generalized kinematic excitation vector, whose elements are defined by the road irregularities and velocity of crossing over road bumps $v(x,t)\rightarrow h(t)$. The damping properties for the system cannot be predicted in the same way as mass and stiffness. Although an approximation for the damping properties can be made by introducing a proportional damping matrix, the mode shapes obtained from such a model are still the same as the undamped ones. Now that the natural frequencies and mode shapes can be found as shown in (3), the viscous damping matrix **B** is directly define as a linear combination of the mass and stiffness matrix as follows:

$$\mathbf{B}=\alpha\mathbf{M}+\beta\mathbf{K} \tag{4}$$

Applying the modal matrix congruential transformation to **B**, the effective modal damping factor can be found as

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_{ri}} + \beta\omega_{ri}\right) \tag{5}$$

Least Squares Complex Exponential Method

LSCE method is a time domain modal analysis method. It explores the relationship between the impulse response function (IRF) of an MDOF system and its complex poles and residues through a complex exponential [5]. A complete development of this method is found in [6]. The LSCE method begins with the transfer function of a linear, viscously damped MDOF system given as

$$\alpha_{ij}(s) = \sum_{r=1}^{n}\left(\frac{A_{ij,r}}{s - s_r} + \frac{A^*_{ij,r}}{s - s^*_r}\right) \tag{6}$$

taking r > n: ${}_rA_{ij} = {}_rA^*_{ij} : s_r = s^*_r$ in equation (5) and applying the inverse Laplace Transform to (6) and assuming that the resulting IRF is sampled at a series of equally spaced time intervals $k\Delta$ (k = 0, 1, …, 2n), then we will have a series of sampled IRF data, where $z_r^k = e^{s_r k\Delta}$:

$$h_k(k\Delta) = \sum_{r=1}^{2n} A_{ij,r} z_r^k \tag{7}$$

In fact at the end of the experiment, we have only this discrete data. In the current work, this data will be used to find the modal parameters of the system [3]. To do this, the next step is to estimate the roots and residues from the sampled data. Mathematically, this means that z_r are roots of a polynomial with real coefficients as shown in equation (8) below:

$$\beta_0 + \beta_1 z_r + \beta_2 z_r^2 + \ldots \beta_{2n-1} z_r^{2n-1} + \beta_{2n} z_r^{2n} = 0 \tag{8}$$

This method was developed by *Gaspard Riche de Prony* in 1795. By using this method the poles of the system can be found. Multiplying both side of equation (8) with a corresponding coefficient β and adding all equalities together, this lead us to a simple relationship between the coefficient β and the IRF sample (9),

$$\sum_{k=0}^{2n} \beta_k h_k = 0 \tag{9}$$

Using equation (9) the hankel matrix can be constructed, from which β can be found easily by using matrix inversion. Now with β coefficients known the roots z_r of equation (8) can be solved easily. Since the complex natural frequencies s_r are determined by the undamped natural frequencies ω_r and damping ratios ξ_r, the natural frequency and damping ratio of the r*th* mode can be derived as follows;

$$\omega_r = \frac{1}{\Delta}\sqrt{\mathrm{In} z_r \mathrm{In} z_r^*} \tag{10}$$

$$\xi_r = \frac{-\mathrm{In}\left(z_r z_r^*\right)}{2\omega_r \Delta} \tag{11}$$

Since the roots z_r of polynomial (8) are known, the mode shape of the system from the IRF can be found easily using equation (7).

Eigensystem Realization Algorithm

The eigensystem realization algorithm is another time domain method for modal parameter identification, more details can be found in [4], [6]. Consider a *N* degree-of-freedom linear dynamic system represented in state-space form at discrete time t = *kΔT*, k = 0, 1, 2, …, with a constant sampling time *ΔT*, as

$$x(k+1) = \mathbf{A}x(k) + \mathbf{B}u(k)$$
$$y(k) = \mathbf{C}x(k) + \mathbf{D}u(k) \qquad (12)$$

where $x(k)$ - is an *n*-dimensional state vector, **A** – state matrix, **B** – input matrix, **C** – output matrix and **D** – feed-through matrix. The essence of ERA is to apply the SVD on the measured impulse response data in order to determine the number of prominent DOFs of a structure and to determine the system matrix **A**, input matrix **B** and output matrix **C** in the state space representation of the system. The system matrix **A** is can then be used for an eigenvalue solution to determine the modal parameters of the system under consideration as follows;

$$\boldsymbol{\psi}^{-1}\mathbf{A}\boldsymbol{\psi} = \mathbf{Z} \qquad (13)$$

The eigenvalue solution of matrix **A** provides the system modal properties. Here, $\boldsymbol{\psi}$ contains 2*N* complex mode shapes. The eigenvalues in the diagonal matrix **Z** are related to the system natural frequencies and damping ratios as follows,

$$Z_r = e^{\lambda_r \Delta} = e^{\left(\lambda_r^R + i\lambda_r^I\right)\Delta} \qquad (14)$$

$$\omega_r = \sqrt{\left(\lambda_r^R\right)^2 + \left(\lambda_r^I\right)^2} \qquad (15)$$

$$\xi_r = -\frac{\lambda_r^R}{\sqrt{\left(\lambda_r^R\right)^2 + \left(\lambda_r^I\right)^2}} \qquad (16)$$

Results and discussion

Two types of analysis were performed in this paper, firstly the analysis was done to a simulated analytical data set with known properties and secondly, these artificial data were processed in MATLAB environment. The simulated analytical data is very useful because the exact values are known. The impulse and frequency response methods were used as verification tools to compare with the identified natural frequencies, damping ratios and mode shapes obtained by using the LSCE and ERA methods. The classical approach to mechanical and structural dynamics was used to obtain the theoretical dynamic properties of the system considered herein. Theoretical values were based on the on the material properties of the system considered in this work.

Figure 2 shows a subplot of the impulse response function (IRF) of LSCE and ERA compared to analytical solution. Shown on the same figure is a close up look of the impulse response function for time t = 4s to t = 6s. The regenerated signal was estimated using the estimated modal parameters, see *Table* 1 below. *Figure* 3 shows the frequency response function (FRF) of the 3DOF mechanical system. The LSCE and ERA estimated FRF closely follows the analytical response over most of the run, and the agreement is excellent both in terms of magnitude and frequency content.

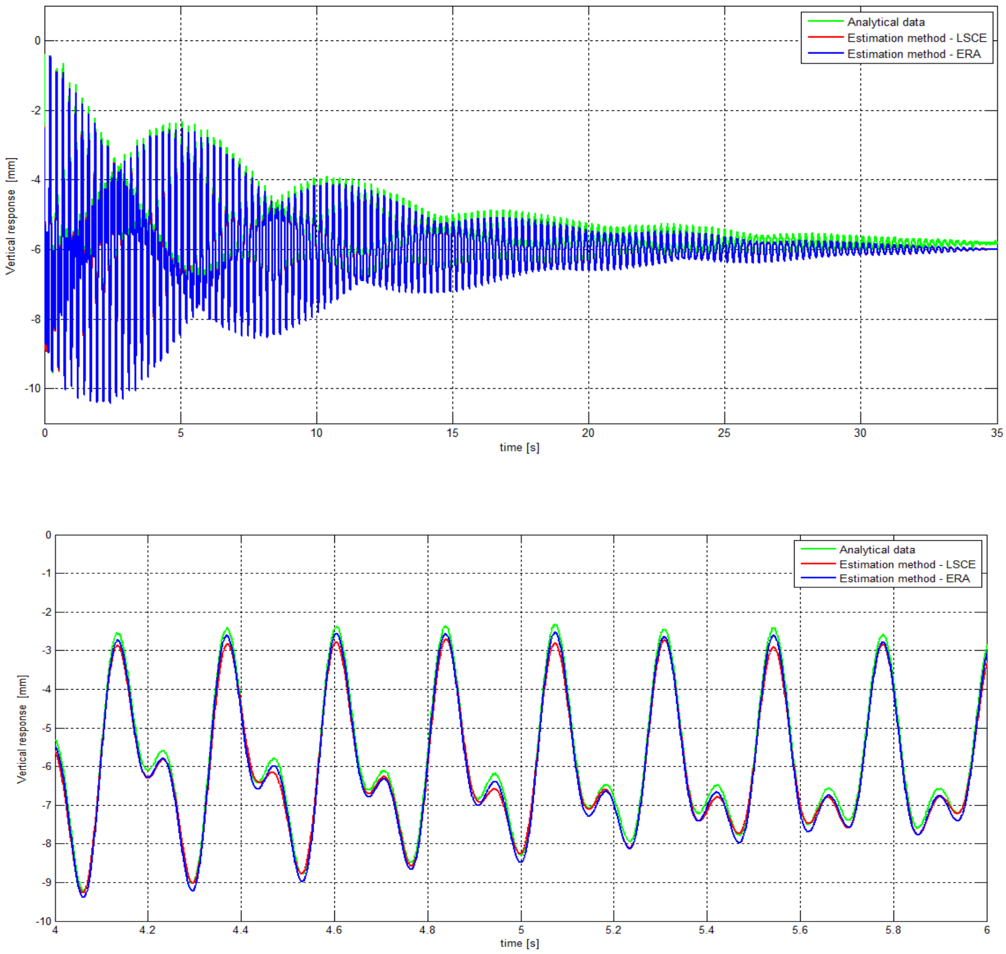

Figure. 2 - Impulse response function (IRF)

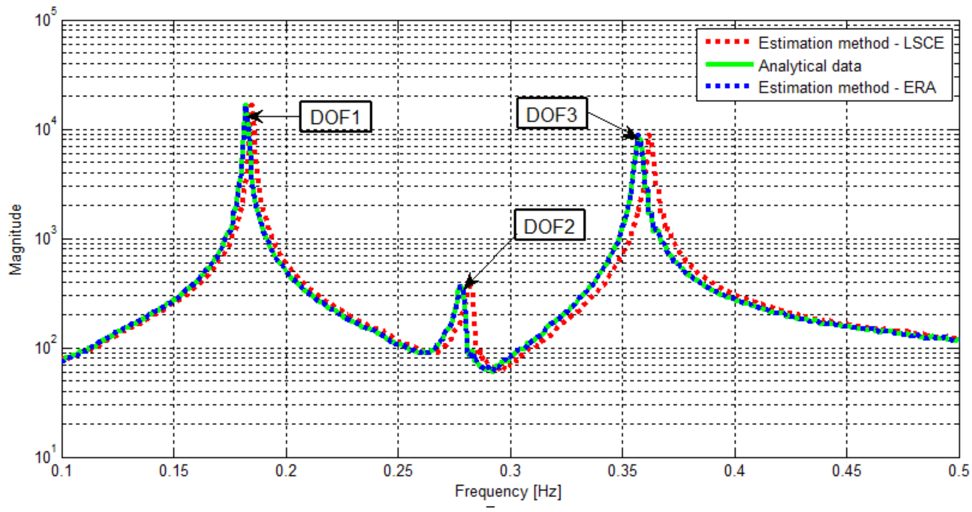

Figure. 3 – Frequency response function (FRF)

Table 1: Analytical and Estimated Modal Parameters

Natural frequencies [Hz]		
Artificial analytical data	ERA	LSCE
0.75393545701061	0.749677111338569	0.70539758026719
1.92544551122749	1.67952032059237	1.80710111688182
10.6662181521164	10.719210669725	11.1255969922537
Damping ratio [-]		
0.530400179630034	0.632022430559199	0.63803495718154
0.590794634180329	0.696506134675492	0.78197869359923
0.765882352445369	0.875185791298039	0.94275392267018
Curve fit standard deviation		
	FRF	412.746205221018
	IRF	0.48944129257183

Conclusion

Modal analysis techniques are generally used to obtain mathematical model of mechanical and structural systems. This paper provided the investigation of two modal analysis techniques and algorithms. The LSCE and ERA algorithms were reviewed, and after examining the theoretical background of these algorithms, application and verification of these methods was done in MATLAB environment. The results showed an acceptable level of agreement between the LSCE, ERA estimated IRF and FRF, in comparison to the artificial analytical data.

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