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Design concept of sandwich beams deflection

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Abstract. The aim of this paper is to introduce a framework for designing simple sandwich beams. Such sandwiches are assumed to be composed of multiple interconnected individual sub-beams made of various materials, having different cross-sectional profiles and different lengths. The work presented in this paper offers a simple framework for evaluation of the deflection of sandwich beams, with application to solution of cantilever sandwich beam composed of arbitrary number of sub-beams of unequal length.

Introduction

The sandwich beams are not new in principle, but the recent development of material science and technology has brought new ideas and possibilities to their practical implementations. Thus also naturally the attempts to develop a simple methods for their design got an important impulse. The concept itself is inspired by a similar approach previously seen e.g. in leaf springs, multi-chamber beams (aluminum/plastic) or in sandwich composite structural elements.

The individual beams are aligned and connected in parallel to each other in order to assure that they deform together, following the same elastic deflection curve. As an example let's consider a cantilevered beam shown in the Fig. 1.

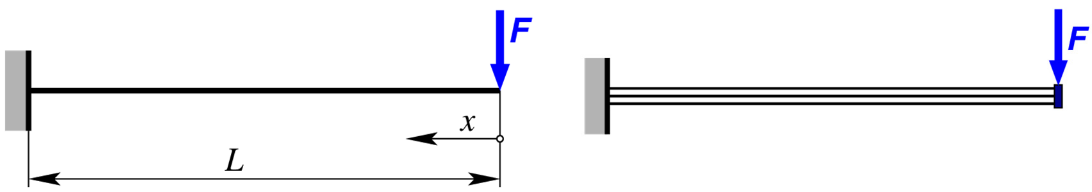

Fig. 1: Cantilever beam scheme (left) and its sandwich version composed of three parallel beams (right).

The beam global load (shear *T(x)* and moment *M(x)*) distribution is independent to the internal structure (composition) of the whole beam.

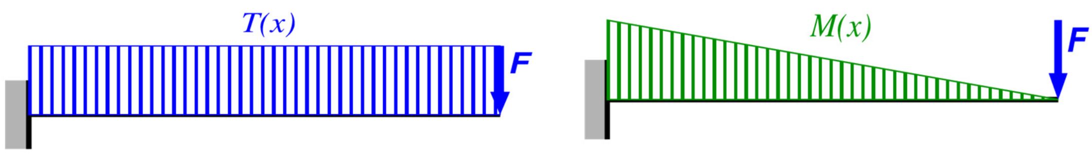

Fig. 2: The shear force (left) and bending moment (right) distribution along the beam.

The (sandwich) internal structure of the beam results into a redistribution of the loading among the individual beams, while keeping their deformation identical for all individual sub-beams of the sandwich group. In the simplest case, assuming that only the shear force is transferred between the individual beams at the point where they are interconnected, shear load is internally redistributed to individual sub-beams depending on their stiffness, while respecting the deformation condition of equal deflection for all sub-beams in the sandwich. The situation is analogical, from the elasticity point of view, to springs attached in parallel and jointly loaded by a force [1,2]. See the illustration in Fig. 3

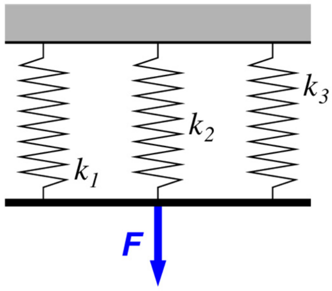

Fig. 3: Springs attached in parallel have a joint equivalent stiffness $k=k_1+k_2+k_3$.

In the same way the total stiffness of the whole sandwich beam can be assumed as an algebraic sum of the stiffnesses of the individual beams. E.g. the sandwich beam depicted in the Fig 1 (right), composed of three individual beams, can be represented by a simple beam with equivalent stiffness (flexural rigidity) $EI=E_1I_1+E_2I_2+E_3I_3$. The material's modulus of elasticity E_i and beam's profile moment of inertia I_i can be different for each individual sub-beam.
Remark: In the case of the mechanical spring analog shown in the Fig. 3 the total force F is redistributed to individual sprigs 1, 2, and 3 proportionally to their stiffness as $F_1:F_2:F_3=k_1:k_2:k_3$, where $F=F_1+F_2+F_3$.

Deflection of a beam

Single sandwich cantilever beam
Assuming that the cantilever beam is composed of a single sandwich, where all the sub-beams have the same length (like in the Fig. 1 - right), the joint equivalent stiffness can be directly used to determine the beam deflection, i.e. the beam elastic curve equation $v(x)$.

$$\frac{d^2v}{dx^2}(x) = \frac{M(x)}{EI} \quad \text{where} \quad M(x) = F \cdot x \tag{1}$$

$$\frac{d^2v}{dx^2}(x) = \frac{F \cdot x}{E_1I_1+E_2I_2+E_3I_3} = \frac{F}{EI} \cdot x \tag{2}$$

From this second order differential equation, the elastic curve can be obtained by double integration [3,4].

$$\frac{dv}{dx}(x) = \frac{F}{EI}\left(\frac{x^2}{2} + C_1\right) \tag{3}$$

$$v(x) = \frac{F}{EI}\left(\frac{x^3}{6} + C_1x + C_2\right) \tag{4}$$

The integration constants C_1, C_2 can be obtained from the boundary conditions at the fixed end of the beam, i.e. at $x=L$, where $v(L) = 0$ and $\frac{dv}{dx}(L) = \varphi(L) = 0$. The displacement and slope of the elastic curve can be finally expressed as:

$$\frac{dv}{dx}(x) = \frac{F}{2EI}(x^2 - L^2) \tag{5}$$

$$v(x) = \frac{F}{6EI}(x^3 - 3L^2x + 2L^3) \tag{6}$$

Remark - on practical realisation of sandwich beams: The sandwich principle can be used to build custom beams with wide range of cross-sectional profiles, allowing to mechanically join together different materials. There exist infinitely many technical realisations of such sandwich beams. As an example, some profiles of sandwich beams composed of simple beams with standard profiles are shown in the Fig. 4.

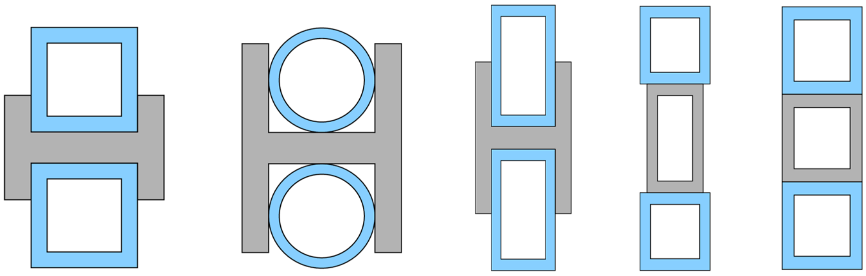

Fig. 4: Example of several sandwich beam profiles built of three individual sub-beams.

Multiple fields cantilever sandwich beam

Besides of building beams with custom profiles, the sandwich construction can be used to reinforce the beams in parts, where the load is higher. In this way the stress can be better redistributed among the profiles in the sandwich, leading to optimized material use and possibly lower mass of the whole beam for given loading forces. In this context, as an example, considering the moment diagram in the Fig. 2, it seems natural to build a sandwich beam with only one beam at the tip (where the moment is small) and to add reinforcement beams while going towards the wall, where the moment attains its maximum value. A schematic view of such sandwich beam with three individual sub-beams of unequal length is shown in the Fig. 5.

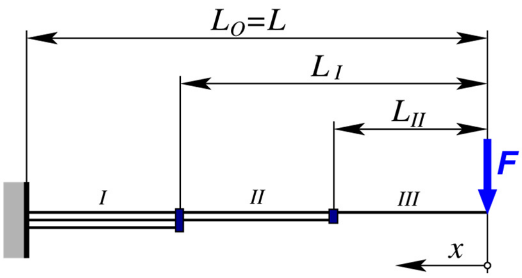

Fig. 5: Sandwich beam with three individual sub-beams and three loading fields.

The deflection of the beam shown in the Fig. 5 can be obtained by applying the above described double integration of the differential equation (1), where *E(x)* and *I(x)* are considered variable (piece-wise constant), with possible jumps in places where the number of sub-beams is changing. E.g. in the field *I*, $(EI)_I = E_1I_1+E_2I_2+E_3I_3$, in the field *II*, $(EI)_{II} = E_1I_1+E_2I_2$, while in the field *III*, $(EI)_{III} = E_1I_1$. The deflection of the beam is solved from the fixed end (where boundary conditions are known), sequentially field by field, by taking the displacement and the slope of the elastic curve at the end of one field as the initial condition for the next field.

Considering the cantilever sandwich beam depicted in the Fig. 5, we can write (according to (3) and (4)) the displacement and slope of the elastic curve in the field I as:

$$\frac{dv_I}{dx}(x) = \frac{F}{(EI)_I}\left(\frac{x^2}{2} + C_{1,I}\right) \tag{7}$$

$$v_I(x) = \frac{F}{(EI)_I}\left(\frac{x^3}{6} + C_{1,I}x + C_{2,I}\right) \tag{8}$$

which is subject to boundary conditions at $x=L$, where $v(L) = 0$ and $\frac{dv_I}{dx}(L) = 0$, leading to integration constants

$$C_{1,I} = \frac{-L^2}{2} \qquad \text{and} \qquad C_{2,I} = \frac{L^3}{3} \quad . \tag{9}$$

The elastic curve in the field II can be obtained from the equations analogical to (7), (8), i.e.

$$\frac{dv_{II}}{dx}(x) = \frac{F}{(EI)_{II}}\left(\frac{x^2}{2} + C_{1,II}\right) \tag{10}$$

$$v_{II}(x) = \frac{F}{(EI)_{II}}\left(\frac{x^3}{6} + C_{1,II}x + C_{2,II}\right) \ , \tag{11}$$

but using different boundary conditions. The smooth transition of the elastic curve between the fields I and II is guaranteed provided that in the transition point located $x = L_I$ the following conditions on the displacement and slope are met:

$$v_{II}(L_I) = v_I(L_I) \qquad \text{and} \qquad \frac{dv_{II}}{dx}(L_I) = \frac{dv_I}{dx}(L_I) \quad . \tag{12}$$

These conditions lead to the integration constants for the second field in the form:

$$C_{1,II} = \frac{1}{2}\left(\frac{(EI)_{II}}{(EI)_I}(L_I^2 - L^2) - L_I^2\right) \tag{13}$$

$$C_{2,II} = \frac{1}{6}\left(\frac{(EI)_{II}}{(EI)_I}(L_I^3 - 3L_I L^2 + 2L^3) - L_I^3\right) - C_{1,II}L_I \tag{14}$$

The integration constants $C_{1,II}$ and $C_{2,II}$ can be then plugged in to the (10) and (11) to obtain explicit formulas for the elastic curve displacement and slope. It is good to note here, that setting $(EI)_{II} = (EI)_I$ leads to $C_{1,II} = C_{1,I}$ and $C_{2,II} = C_{2,I}$, meaning that the elastic curve solution in the field II simply keeps following the equation from the previous field I, without any change. The same approach can be used to continue by prolongation of the solution to the field III.

Generalisation to beams with arbitrary number of fields

In fact, the above described procedure can easily be generalized to a sandwich beam with arbitrary number *N* of sub-beams (and fields). In such a case, the first field is solved exactly as described, while for any consequent field with index *K+1,* the solution can be obtained from a recurrent formulas generalizing (12), resp. (13) and (14). Considering the smooth transition from the field *K* to *K+1*, in the transition point located $x = L_K$ the following conditions on the displacement and slope have to be satisfied:

$$v_{K+1}(L_K) = v_K(L_K) \qquad \text{and} \qquad \frac{dv_{K+1}}{dx}(L_K) = \frac{dv_K}{dx}(L_K) \tag{15}$$

These conditions lead to the integration constants for the *K-th* field in the form:

$$C_{1,K+1} = \frac{1}{2}\left(\frac{(EI)_{K+1}}{(EI)_K}\left(L_K^2 + 2C_{1,K}\right) - L_K^2\right) \tag{16}$$

$$C_{2,K+1} = \frac{1}{6}\left(\frac{(EI)_{K+1}}{(EI)_K}\left(L_K^3 + 6C_{1,K}L_K + 6C_{2,K}\right) - L_K^3\right) - C_{1,K+1}L_K \tag{17}$$

The recurrent formulas (16) and (17) for integration constants $C_{1,K+1}$ and $C_{2,K+1}$ can be used repeatedly for *K=I,…,N-1* and substituted to the formulas analogical to (10), (11)

$$\frac{dv_{K+1}}{dx}(x) = \frac{F}{(EI)_{K+1}}\left(\frac{x^2}{2} + C_{1,K+1}\right) \tag{18}$$

$$v_{K+1}(x) = \frac{F}{(EI)_{K+1}}\left(\frac{x^3}{6} + C_{1,K+1}x + C_{2,K+1}\right) , \tag{19}$$

providing the solution for elastic curve for $x \in \langle L_{K-1}; L_K \rangle$.

Here we have assumed that each of the *N* sub-beams has constant stiffness (flexural rigidity) along its length, which results into a general expression for the equivalent stiffness of the *K-th* field

$$(EI)_K = \sum_{i=1}^{N-K+1} E_i I_i \quad . \tag{20}$$

Remark: Also in this case it can be assumed that the loading shear force *T(x)* redistributes in each field to individual sub-beams, forming the sandwich, proportionally according to their stiffness $(E_i I_i)$. It means that, exactly as in the mechanical spring analog depicted in the Fig. 3, the deformation of all sub-beams is the same, while the loading force is redistributed, with higher load being transferred to the stiffer sub-beams.

Conclusions

The presented method of solution of the deflection of sandwich beams results into simple way of obtaining closed form formulas for elastic curve of the beam. This is in contrast to the recent trends in simulation of the non-trivial beams, where only numerical representation of the solution can be obtained. Despite of this trend, the presented analytical method can be used in providing fast design concepts solutions in the initial stages of the complex mechanical devices.

It should be pointed out and understood that the presented approach is simplified and it heavily depends on the assumption that only the shear forces are transferred from one sub-beam. The practical situation can differ from the described one depending especially on the way how the sub-beams are interconnected in the sandwich. The specific case considered here however represents the worst case scenario, which justifies its use in the preliminary design calculations.

The future extensions of the presented work will focus on the following topics:

- *validation* – The presented analytical approach can (and should) be confronted with experimental results, or at least with more accurate (e.g. finite-element) numerical solutions.
- *extension* – The examples presented here were based on cantilever beam loaded by a single force at its free end. Both the loading and the type (shape) of the beam can be modified and generalized within the presented framework.
- *implementation* – There arised several questions concerning practical implementation possibilities of the sandwich beams. Among them namely the methods of interconnection of sub-beams, treatment of contact between the beams, material and structural properties of sub-beams.
- *applications* – The motivation for the presented study came from solving practical case of a moving arm of a mechanism. There are however many other application areas, including e.g. biomechanics or civil and transport engineering.

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