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IDENTIFICATION OF ORTHOTROPIC HYPERELASTIC MATERIAL PROPERTIES OF CORD-RUBBER CYLINDRICAL AIR-SPRING

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In this paper we present an attempt to identify experimentally the coefficients of strain energy function of the hyperelastic orthotropic material of the thin cylindrical air-spring. The components of the deformation gradient are determined from measured displacements of the rectangular grid drawn on the cylindrical surface of the spring. The true Cauchy stress tensor is calculated from the membrane theory. The deformed shape of the spring surface is determined from the photographic records. The strain energy function is expressed in terms of tensorial invariants with regard to the assumed material symmetry. The coefficients are determined by means of the nonlinear least squares method.

Key words: Hyperelasticity, Cord-reinforced rubber material, Orthotropic constitutive equation

1 Introduction

The linear constitutive relations valid for small strain linear regime can be sometime extended to the large displacement small strain case by rewriting them as relationship between the Lagrangian strain tensor and the second Piola - Kirchhoff stress tensor. This way can however result in unrealistic stresses when the large strains are present. Elasticity in the fully nonlinear range is established in terms of a hyperelastic strain energy potential and different isotropic strain energy functions are implemented in many finite elements codes, some of them even allow to users to incorporate their own material law.

The strains of our air-spring reach about 20% in the operational pressure interval thus they fall into the finite strain range. We suppose the cylindrical air-spring sheathing is made of rubber reinforced by two families of helically wound fibers so that the mechanical properties of the material are direction dependent. We suppose also that the material of the spring can sustain finite strains without noticeable volume changes.

The development of the constitutive theory of anisotropic elastic or viscoelastic materials at finite strains is still far to be complete and the publications in this field are sparse. The constitutive equations of the transversally isotropic material in the nonlinear stress and deformation domain are presented in the papers of Verron and coll. [1,2], Bonet and Burton [3]. We use the consistent constitutive model of direction dependent hyperelastic material presented in papers of Ogden, Holzapfel, Gasser and coll. [4-7] applied by authors to the problem of the mechanical response of arterial walls and of fiber reinforced composites at finite strains. They use a particular Helmholtz free energy function, which allows modeling the behavior of the orthotropic composite.

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2 Hyperelasticity in orthotropic case

Let $\boldsymbol{X}$ represents the initial position vector of any particle and let $\boldsymbol{x} = \phi(\boldsymbol{X})$ denotes its current position vector. The derivatives of the mapping ϕ define the deformation gradient tensor $\boldsymbol{F}_{ij}$

$$\boldsymbol{F}_{ij} = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{X}_j} \tag{1}$$

The right Cauchy-Green deformation tensor is defined by

$$\boldsymbol{C}_{ij} = \boldsymbol{F}_{ki}\,\boldsymbol{F}_{kj}, \text{ in the matrix notation } \boldsymbol{C} = \boldsymbol{F}^{T}\,\boldsymbol{F}. \tag{2}$$

Let $\boldsymbol{\sigma}_{ij}$ be the Cauchy true stress tensor defined in the deformed configuration and $\boldsymbol{S}_{ij}$ the second Piola-Kirchhoff stress tensor relative to undeformed configuration related by

$$\boldsymbol{S} = J\,\boldsymbol{F}^{-1}\,\boldsymbol{\sigma}\,\boldsymbol{F}^{-T} \tag{3}$$

where J = det $\boldsymbol{F}$ = ρ_0 / ρ is Jacobian of the transformation (for the isochoric deformation J=1).

For a hyperelastic material we take the material properties to be characterized in terms of a strain energy function (per unit volume) *W(**F**)* which must be invariant under change of observer frame of reference. The right Cauchy-Green tensor Cij possess such invariance thus the strain energy function can be written in terms of invariants of this deformation tensor. The the orthotropic symmetry of material restricts the way *W* depends on $\mathbf{C}_{ij}$. After Verron [1,2], Holzapfel and Gasser [6] we define the two families of reinforcing fibers by their orientation vectors $\boldsymbol{a}_1(\boldsymbol{X})$ and $\boldsymbol{a}_2(\boldsymbol{X})$ and we introduce the two orientation tensors

$$\boldsymbol{A}_1 = \boldsymbol{a}_1\,\boldsymbol{a}_1^{T}\,,\ \boldsymbol{A}_2 = \boldsymbol{a}_2\,\boldsymbol{a}_2^{T}\,. \tag{4}$$

We assume the isochoric deformation and we neglect the dissipation due to irreversible effects. The energy stored in the fibers is assumed to be governed by an exponential function. The free energy function can be supposed in the form [6]

$$W(I_1,I_4,I_6) = \frac{c}{2}(I_1 - 3) + \frac{k_1}{2k_2}\{\exp[k_2(I_4 - 1)^2 - 1\} + \frac{k_1}{2k_2}\{\exp[k_2(I_6 - 1)^2 - 1\}, \tag{5}$$

where $I_1(\boldsymbol{C}) = \mathrm{tr}\,\boldsymbol{C},\ I_4(\boldsymbol{C},\boldsymbol{a}_1) = \boldsymbol{C} : \boldsymbol{A}_1,\ I_6(\boldsymbol{C},\boldsymbol{a}_2) = \boldsymbol{C} : \boldsymbol{A}_2,$ (7)

I_1 is the principal invariant of $\boldsymbol{C}_{ij}$ known from the isotropic theory and the invariants I_4 and I_6 associated with the anisotropy caused by the two families of fibers determine the squares of the stretches in the fiber directions. The stress-like material parameters $c>0$, $k_1>0$ and the dimensionless parameter $k_2>0$ must be determined experimentally.

The second Piola-Kirchhoff stress tensor can be calculated by differentiation of the strain energy function with respect to the Cauchy-Green deformation tensor $\boldsymbol{C}_{ij}$

$$\boldsymbol{S}_{ij} = 2\frac{\partial W}{\partial \boldsymbol{C}_{ij}} \tag{8}$$

The components of the true Cauchy stress tensor $\boldsymbol{\sigma}_{ij}$ in the deformed configuration of our thin cylindrical membrane are

$$\sigma_{ij} = \sigma_{\Theta} = p\frac{r_c}{t_c},\ \sigma_2 = \sigma_m = p\frac{r_c^2 - r_p^2}{2r_c t_c} \tag{9}$$

where p is the internal pressure, r_c and t_c denote the radius and the thickness of the membrane in the deformed configuration and r_p is the radius of the piston face.

3 Experimental determination of deformation gradient F_{ij}

The experiments were achieved in Hydrodynamic laboratory of TU of Liberec on the equipment enabling to measure and to record the internal pressure and volume of the spring, the axial and off-axis force and the moment at the piston. The rectangular grid drawn at the surface of the spring is shown in ***Fig***.1. The internal pressure was changed gradually by step 0.05 MPa in the working range of the spring from 0.1 to 0.6 MPa.

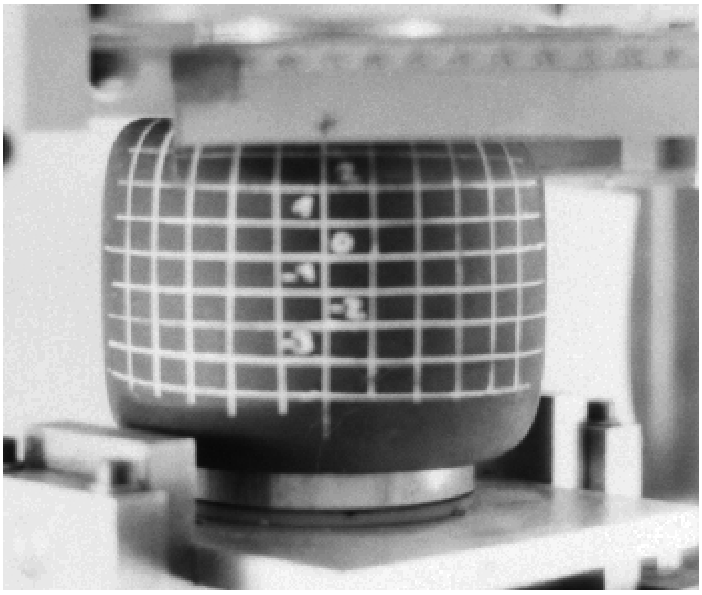

Figure 1. Experimental set up

The separate photographs of the deformed surface were enregistered by the camera. The number of steps in one cycle of loading and unloading was 22. The negatives were scanned by the scanning device. The positions of the grid points with respect to the beginning of reference co-ordinate system were determined from the pictures in bitmap by means of our software *ProLad01*.

The components of the current position vectors of grid points have been fitted by the functions of two variables by means of the linear last squares method and their gradients in the hoop and axial direction were calculated. The component $\boldsymbol{F}_{33}$ of the deformation gradient was determined from the incompressibility assumption according to the expression

$$F_{33} = A_0 / A_c\ , \tag{10}$$

where A_0 and A_c denote the area of the grid fields in the reference and current position respectively. The calculated extradiagonal components of the deformation gradients were on order of two less than the diagonal ones and were neglected in the further calculations then we suppose the circumferential and the axial direction are the principal ones.

The dependence of the deformation gradient components on the loading pressure is shown in ***Fig***. 2. The full lines and the single points denote the values calculated from the linear and quadratic regression functions respectively. The curves show the slight hysteresis.

The diameter of the cylindrical surface of the spring at each loading stage was determined from the records. The ratio of the thickness in the deformed and current configuration is assumed to be

$$t_c / t_0 = A_0 / A_c = 1 / F_{33} \tag{11}$$

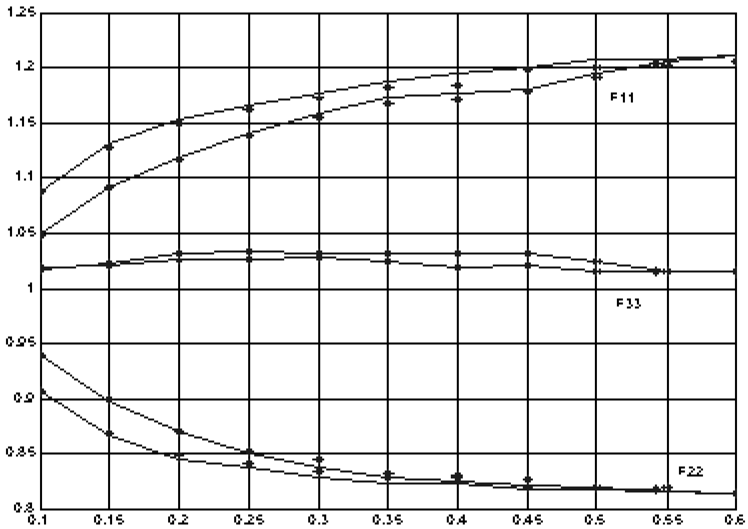

Figure 2. Components of deformation gradient ***F***

4 Determination of material parameters

The components of the Cauchy true stress tensor at the each stage of loading were determined from the experimental results according to the relations (9) then the components of Piola-Kirchhoff stress tensor were calculated from the relations (3) and the components c_{11} and c_{22} of the right Cauchy-Green deformation tensor were calculated from the deformation gradient after the relation (2). We suppose the isochoric deformation and the third component $c_{33} = 1 / (c_{11}\, c_{22})$.

We suppose the reinforcing f ibers are double-helically arranged in the matrix material symmetrically to the circumferential direction. The angle α of fibers is supposed to be 39°. In such case the orientation tensors (4) are identical and the invariants I_4 and I_6 as well and the free energy function takes the form

$$W(I_4, I_6) = \frac{c}{2}(I_1 - 3) + \frac{k_1}{k_2}\left\{\exp\left[k_2 (I_4 - 1)^2\right] - 1\right\}. \tag{5a}$$

The components of the Piola-Kirchhoff stress tensor are exprimed after the relation (8) as

$$\begin{aligned} \boldsymbol{s}_{11} &= c\Big(1 - \frac{1}{c_{11}^2 c_{22}}\Big) + 4k_1 \cos^2 \acute{a}\; f(\boldsymbol{C}, \boldsymbol{A}) \exp[k_2\, f^2(\boldsymbol{C}, \boldsymbol{A})] \\ \boldsymbol{s}_{22} &= c\Big(1 - \frac{1}{c_{11} c_{22}^2}\Big) + 4k_1 sin^2 \acute{a}\; f(\boldsymbol{C}, \boldsymbol{A}) \exp[k_2\, f^2(\boldsymbol{C}, \boldsymbol{A})] \end{aligned}, \tag{8a}$$

where the anisotropic function $f(\boldsymbol{C}, \boldsymbol{A})$ has the form

$$f(C,A) = c_{11}\cos^2\alpha + c_{22}\sin^2\acute{a} - 1\,. \tag{8b}$$

The system (8a) of 42 nonlinear equations was solved by means of the different nonlinear least squares methods of the optimization toolbox in MATLAB code.

The resulting functions of the right Cauchy-Green deformation tensor shown in the following figures are plotted with respect to the circumferential component c11 on x-axis and the axial component c_{22} on y-axis. The ***Fig***. 3 shows the experimental values of the Piola-Kirchhoff stress tensor components plotted as points on the surface of the energy function gradients calculated according to the fitted values of the material parameters c = 2.63 *MPa*, k_1 = 18.8 *MPa*, k_2 = 13.4. It is apparent that the parameters of the energy function appropriate to our material are fitted with the acceptable probability.

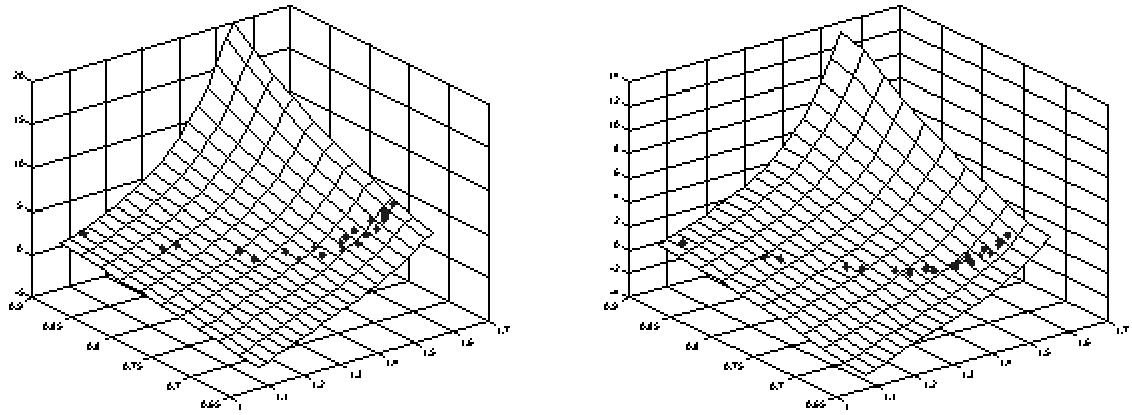

Figure 3. Piola-Kirchhoff components- circumferential in the left, axial in the right

The free energy function (5a) for the fitted parameters and its contour lines are shown in the **Fig**. 4. The free energy function should have strict local minimum at point (1,1). As can be seen from the figures the energy potential is convex and anisotropic.

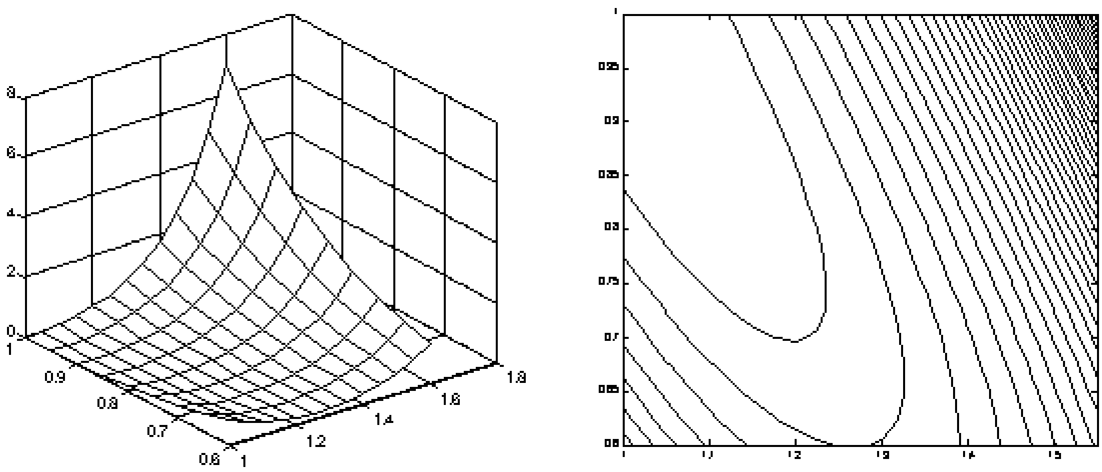

Figure 4. Strain energy function and its contour lines in the experimental range

5 Conclusion

The problem of the identification of the material parameters was solved. The proposed strain energy function will be implemented into FEM calculus of states of deformation of our cylindrical air-spring. When the fitted material parameters should be used in a prediction of states of deformation these should not be far from the range of deformation for which the experimental tests were conducted. The two-dimensional formulation used in the parameter fitting does not, in general, permit the stress response under certain combined loading (such as inflation and torsion) to be modeled. However such formulation, because it omits c12, c13 and c33 is inherently limited to specific kinematics or to the membrane description. The two-dimensional anisotropic energy function may be valuable under some conditions, but it should be used with caution. This approach, which is concerned mainly with fitting the constitutive equations to the experimental data, is not capable of relating the deformation mechanism to the known architectural structure of the membrane wall. The material parameters have no direct physical meaning and are therefore treated as numbers without clear physical interpretation.

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References

[1] Chevaugeon, N., Verron, E., Peseux, B., Finite element analysis of nonlinear transversely isotropic hyperelastic membranes for thermoforming applications, Proc. of Europ. Congr. on Comput. Meth. in Appl. Sci.& Engrg. ECCOMAS 2000, Barcelona

[2] Marckmann, G., Verron, E., Peseux, B., B-splines modeling of free inflation of cylindrical rubberlike membranes, Proc. of Europ. Congr. on Comput. Meth. in Appl. Sci.& Engrg. ECCOMAS 2000, Barcelona

[3] Bonet, J., Burton, A.J., A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations, Comput. Methods Appl. Mech. Engrg. 162, 1998, 151-164

[4] Ogden, R.W., Schulze-Bauer, C.A.J., Phenomenological and structural aspects of the mechanical response of arteries, Proc. in Mechanics in Biology, J.Casey & G. Bao eds., AMD-Vol. 242, BED-Vol. 46, N.Y., 2000, 125-140

[5] Holzapfel, G.A., Gasser, T.C., Ogden, R.W., A new constitutive framework for arterial wall mechanics and a comparative study of material models, J. of Elasticity, Nov. 23, 2000

[6] Holzapfel, G.A., Gasser, T.C., A viscoelastic model for fiber-reinforced composites at finite strains: Continuum basis, computational aspects and applications, Biomech Preprint series of Graz University of Technology, http://www.cis.tugraz.at/biomech/papers.htm, to appear in Computer Methods in Applied Mechanics and Engineering

[7] Holzapfel, G.A., Stadler, M., Ogden, R.W., Aspects of stress softening in filled rubbers incorporating residual strains, in: A. Dorfmann & A. Muhr eds, Constitutive Models for Rubber, Balkema, Rotterdam, 1999, 189-193

[8] B. Marvalová, Experimentální určení elastických vlastností materiálu válcové pryžové pneumatické pružiny, Proc. of VIII. Int. Conf. on the Theory of Machines and Mechanisms, IFTOM, Sept. 2000, Liberec, 431- 434