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Experimental Stress Analysis

Everything you always wanted to know about stress but were afraid to ask

Miloslav Okrouhlík[1]

Abstract: The author ponders about meaning of stress and of other mechanical variables that are consensually defined and cannot be grasped, described and/or measured directly.

Keywords: Stress, Force, Time, Continuum mechanics, Terminology and meaning.

1. Introduction

The term ***stress*** in current communication is understood differently from the way it is used in mechanical engineering practice. The Cambridge International Dictionary offers for the item ***stress*** the following: *Great worry caused by a difficult situation* or *a force that acts in a way which tends to change the shape of an object*. Among many examples from the same source let's quote the one which might be considered amusing in our community, i.e. *Yoga is a very effective technique for combating stress*. Often the ***stress*** is being considered to be almost equivalent to *strain* as in: *Many joggers are plagued by knee stress and foot strain caused by unsuitable footwear*. Other sources offer similar explanations. An example taken from Wikipedia: *We generally use the word '**stress**' when we feel that everything seems to have become too much - we are overloaded and wonder whether we really can cope with the pressures placed upon us. **Anything that poses a challenge or a threat to our well-being is a stress**. Some stresses get you going and they are good for you - without any stress at all many say our lives would be boring and would probably feel pointless. However, when the stresses undermine both our mental and physical health they are bad. In this text we shall be focusing on stress **that is bad for you**.* In this paper, in contradistinction to the previous example that might invoke a gloomy mood in reader's mind, we will concentrate on meanings ***that are good to you***, i.e. on **mechanical stress** (Spannung in German, contrainte in French, napětí in Czech). The IFToMM (International Federation for the Promotion of Mechanism and Machine Science) online dictionary gives a more acceptable explanation for the ***stress***, i.e.: *Limits of the ratio of force to the area it acts, as the area tends to zero*. The definition of stress, being presented this way, however, says nothing about the distribution of the force 'above' the mentioned area. Furthermore, the mentioned dictionary defines ***stress*** by introducing a new term, namely ***force*** that is, in turn, specified as an *action* there, i.e.: *Action of its surroundings on a body tending to change its state of rest or motion*. Evidently a definition from the pen of a rigid body person. Other ***force*** definitions appearing in solid mechanics textbooks are not more comprehensive either and describe ***force*** rather circularly by its effects. A few examples are presented here. In Encyclopaedia of Physics, Vol. III/1, Edited by Fluege, Springer, Berlin, 1960 on page 532 one finds an alleged 'dAlambert's quotation, i.e.: ***Force** is only a name for the product of acceleration by mass*. Similarly in the Theory and Problems of Continuum Mechanics by Mase, G.R., Schaum's Outline Series, Mc Graw Hill, 1970 one

[1] *Prof. Ing. Miloslav Okrouhlík CSc, Institute of Thermomechanics, Academy of Sciences of the Czech Republic, Dolejskova 5, 182 00 Prague 8, The Czech Republic, ok@it.cas.cz*

finds: ***Forces** are vector quantities which are best described by intuitive concepts as push or pull*.

In terms of proper and clear definitions the mechanical variables *force* and *stress* can be compared to the definition of *time*. St. Augustine in his Book 11 of Confessions ruminates on the nature of time, asking: *What then is time? If no one asks me, I know: if I wish to explain it to one that asketh, I know not*[2].

So both time and stress (and force and other variables in mechanics not mentioned here) are defined consensually. We understand them rather intuitively; we might have problems to measure them directly, which – however – does not prevent us to purposefully use them in engineering practice regularly. No one would ever have a tendency to challenge them.

2. Symmetry of stress tensor

*In continuum mechanics, **stress** is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, and result in deformation of the body's shape.*

This nice definition, taken from http://en.wikipedia.org/wiki /Stress_(mechanics), we could gladly accept and start with.

It was Claude-Luis Navier (1785 – 1836) who for the first time formulated equations of motions for a generic point of a continuous body. Augustin-Luis Cauchy (1789 – 1857) accepted the stress definition introduced by Adhémar Jean Claude Barré Saint-Venant (1797 – 1886) and generalized and extended his ideas. For more details see [1].

In Oeuvres complètes d'Augustin Cauchy. Série 2, tome 8 from 1828 is Cauchy's contribution titled SUR LES EQUATIONS QUI EXPRIMENT LES CONDITIONS D'ÉQUILIBRE OU LES LOIS DU MOUVEMENT INTÉRIER D'UN CORPS ÉLASTIQUE OU NON ÉLASTIQUE[3] where he defines the force quantities and writes them in the form

$$\begin{matrix} A & F & E \\ F & B & D \\ E & D & C \end{matrix}.$$

Today we would call them the components of stress tensor. It is worth noticing that from the very beginning Cauchy assumes that the stress components for a material point have a symmetric character[4]. In Cauchy's notation his famous equations of equilibrium have the form

[2] *Quid est ergo tempus? Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio.* The quotation in Latin is taken from www9.georgetown.edu/faculty/jod/latinconf/11.html.
[3] See http://gallica.bnf.fr/ark:/12148/bpt6k90200c.image.f4.langEN.
[4] … pour corps solides considérés comme des systèmes de points materiels distincts les uns des autres, mais séparés par les distances très petites … he thus considers distinguishable material points with zero volumes, and thus infinite density, that are very close together. Here, the limit approach from an elementary volume to a point is not yet considered.

$$\frac{\partial A}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial E}{\partial z}+\rho X=0,$$
$$\frac{\partial F}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial D}{\partial z}+\rho Y=0,$$
$$\frac{\partial E}{\partial x}+\frac{\partial D}{\partial y}+\frac{\partial C}{\partial z}+\rho Z=0.$$

Besides the equilibrium equations he also presents the equations of motion in his text. In today's notation they would read

$$\frac{\partial \sigma_{ij}}{\partial x_j}+\rho b_i=\rho \ddot{x}_i\,. \tag{1}$$

If inertia forces could be neglected, there is zero on the right-hand side and then Eqs. (1) become the equilibrium equations and are equivalent to original Cauchy's formula presented above. It should be reminded that the forces in continuum mechanics are related either to a unit of mass – then they are called **body forces** [N/kg] and denoted b_i or to a unit of volume – then they are called **volumetric forces** [N/m^3] and denoted f_i. The relation between both types of forces is $f_i=\rho b_i$. This allows writing the Cauchy's equations in an alternative form

$$\frac{\partial \sigma_{ij}}{\partial x_j}+f_i=\rho \ddot{x}_i\,. \tag{2}$$

In today's textbooks we generally accept Cauchy's symmetry assumptions. Keeping the standard notation for the rest of the text we could proceed as follows. Deriving geometrical relations between the components of the stress tensor σ_{ij} and the stress vector T_i we consider the material element to be a 3D body – a cube whose orientation in space is uniquely defined by out-pointing normals to its respective walls as shown in Fig. 1.

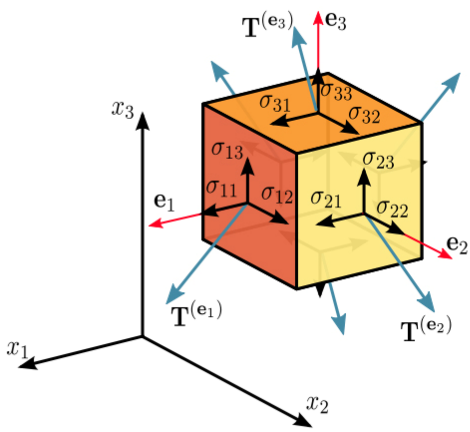

Fig. 1. Material element, components of stress tensor and stress vector

When, however, the equations of motion are being conceived, we suddenly consider the 3D cube as a body of infinitesimal size – approaching zero dimensions in a limit process – and instead of six equations for a body in 3D space we take into account the equations considering the force equilibrium only. Couple equilibrium conditions – under the assumption that the stress levels across the elements' walls are constant and that there are no externally applied couples – are satisfied identically and there is no need to write them explicitely.

Limiting our attention to 3D continuum with small strains and assuming that rigid body rotations can be neglected then there are three Cauchy equations (1), nine kinematic relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

and nine equations representing constitutive relations, i.e.

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \, . \tag{4}$$

All together we have 15 equations for evaluating three displacement components u_i, nine strain components ε_{ij} and nine stress components σ_{ij} – (3 + 9 + 9 = 21). This is, however, enough only if we take the symmetry of stress and strain tensors for granted – (3 + 6 + 6 = 15).

To have a chance to discuss the symmetry of stress tensor let's recall the classical proof of symmetry as it appears in standard textbooks. It is based on conservation of angular momentum and goes as follows.

Denoting the radius vector of a generic material point $r = \{x_1 \; x_2 \; x_3\}^{\mathrm{T}}$, the vector of surface forces t_i, the body forces b_i, then for a continuous body – having volume *V*, surface *S* and density ρ – the conservation of angular momentum can be expressed by

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \left(\vec{r} \times \rho \vec{v}\right) \mathrm{d}V = \int_S \left(\vec{r} \times \vec{t}\right) \mathrm{dS} + \int_V \left(\vec{r} \times \rho \vec{b}\right) \mathrm{d}V \, .$$

By $\vec{v}$ we denote velocity and the scalar quantity t stands for time. Using the Cauchy relation $t_n = \sigma_{jn} e_j$, the Gauss theorem and the relation for the material derivative of momentum $\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho v_i \,\mathrm{d}V = \int_V \rho \frac{\mathrm{D}v_i}{\mathrm{D}t} \,\mathrm{d}V$ we get

$$\int_V \in_{rmn} \frac{\mathrm{D}}{\mathrm{D}t}\left(x_m v_n\right) \rho \,\mathrm{d}V = \int_V \in_{rmn} \left(\frac{\partial \left(x_m \sigma_{jn}\right)}{\partial x_j} + x_m \rho b_n \right) \mathrm{d}V \, .$$

Realizing that $\frac{\mathrm{D}x_m}{\mathrm{D}t} = v_m$, $\frac{\partial x_m}{\partial x_j} = \delta_{mj}$ and $\frac{\partial \left(x_m \sigma_{jn}\right)}{\partial x_j} = \frac{x_m \partial \sigma_{jn}}{\partial x_j} + \delta_{mj}\sigma_{jn} = x_m \frac{\partial \sigma_{jn}}{\partial x_j} + \sigma_{mn}$

we can write $\int_V \in_{rmn} \left(v_m \, v_n + x_m \underbrace{\frac{\mathrm{D}v_n}{\mathrm{D}t}}_{1} \right) \rho \,\mathrm{d}V = \int_V e_{rmn} \left[x_m \underbrace{\left(\frac{\partial \sigma_{jn}}{\partial x_j} + \rho b_n \right)}_{2} + \sigma_{mn} \right] \mathrm{d}V$.

The terms denoted 1 and 2 cancel out, furthermore $\in_{rmn} v_m \, v_n = 0$ and what remains is $\int_V \in_{rmn} \sigma_{mn} \,\mathrm{d}V = 0$. The last relation, independently of the volume considered, is satisfied only if $\sigma_{mn} = \sigma_{nm}$, i.e. if the stress tensor is symmetric. Quod erat demonstrandum – seemingly.

Let's summarize. The described continuum model, in which moment equations are neglected, implicitly assumes that the material particle approaches a point in limit and that the forces between material particles are collinear and opposite. Furthermore, neither external nor body couples act upon the material element. Thus the stress tensor is symmetric only if all these assumptions are satisfied and the presented proof only confirms it. The above reasoning does not prove that the stress tensor is symmetric in general. One might have a temptation to claim that that the stress tensor need not be symmetric in nature. But stress is the conception of human mind – it was not discovered, it was invented.

The notion of stress symmetry, deeply embedded into our minds, is fully acceptable for most of engineering tasks. And of course, the theorem of conjugate shear stresses, claiming that $\sigma_{ij} = \sigma_{ji}$, is based on the stress symmetry.

3. Non-symmetric stress tensor

The continuum model with a non-symmetric stress was considered by Cosserat's brothers. Eugène-Maurice-Pierre Cosserat (1866 – 1931) was a French mathematician and astronomer. He studied at École Normale École Normale Supérieure and later became the director of Toulouse astronomical observatory and a member of French Academy of Sciences. Together with his brother Françoise they studied – among other things – continuum mechanics problems.

When analysing the equilibrium conditions of a material point they considered not only force-stress tensors σ_{ij} expressed in [Pa] but the moment-tensors μ_{ij} in [Pa m] as well. A 2D sketch is in Fig. 2.

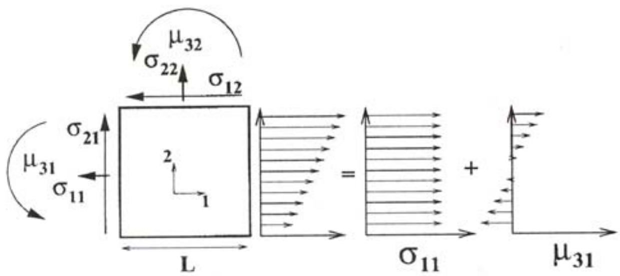

Fig. 2. Force and couple components of stress tensor

It is assumed that the interaction between neighbouring material particles is under way of both the force t_i and the couple vectors m_i. The relation between stress and vector components is provided by an extension of what we today call the Cauchy relation i.e. $t_i = \sigma_{ij}\ e_j$ and $m_i = \mu_{ij}\ e_j$.

Then, both force and couple contributions appear in equations of motion, i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \ddot{x}_i \text{ and } \frac{\partial \mu_{ij}}{\partial x_j} - \in_{ikl} \sigma_{kl} + c_i = I \ddot{\phi}_i .$$

One could notice that besides the volumetric forces f_i there are volumetric couples c_i in equations of motion as well. The variable I denotes rotational inertia. In contradistinction to conventional continuum mechanics, based on symmetric stress tensors, one assumes that the material element has its space orientation which is uniquely prescribed by its translation u_i and rotations φ_i .The Cosserat strains are of two types. One could say the translation strain and the rotary strain that could be expressed followingly

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \in_{ijk} \varphi_k \qquad \text{and} \qquad \kappa_{ij} = \frac{\partial \varphi_i}{\partial x_j} .$$

Until relatively recently the relations describing Cosserat continnum were considered to be an intellectual whim of little practical importance. Now, the Cosserat theory becomes an effective tool for realistic modeling of so called nonlocal problems with materials where particle size effects have to be accounted for as for grains, fibres and cellular structures and where the interaction between particles is provided for not only by forces but by couples as well.

Details concerning the numerical implementation of Cosserat tasks is in the contribution presented by Sluyse and de Borst in [2]

4. Symmetric stress tensors for finite deformation world – true vs. engineering

As mentioned before, in linear elasticity, based on infinitesimal strains and neglected rigid body motions, the stress is defined as a limiting ratio of an elementary force to an elementary area. It is implicitly assumed that the applied force $\Delta^t \mathbf{r}$ belongs to the current (deformed) configuration, say ${}^tC$, while the area $\Delta^0 A$ being loaded belongs to original (undeformed) configuration, say ${}^0C$. This way the engineering stress is defined. When the deformations, strains and rigid body motions are not infinitesimal, then the elementary force has to be related to elementary area $\Delta^t A$ in the deformed configuration, defining thus the true (sometimes called Cauchy) stress. The engineering and true stress vectors for both cases are

$$\mathbf{t}^{\mathrm{eng}} = {}^t_0\mathbf{t} = \lim_{\Delta^0 A \to 0} \frac{\Delta^t \mathbf{r}}{\Delta^0 A} \qquad \text{and} \qquad \mathbf{t}^{\mathrm{true}} = {}^t_t\mathbf{t} = \lim_{\Delta^t A \to 0} \frac{\Delta^t \mathbf{r}}{\Delta^t A} .$$

One should notice that the upper left index associates the quantity to the configuration in which it is 'measured', while the lower left index identifies the configuration to which the quantity is related. Introducing normal vectors in reference and current configurations by ${}^0e_i$ and ${}^te_i$ respectively, and using the Cauchy's relation, the corresponding engineering and true stress tensors (${}^t_0\sigma_{ij}$ and ${}^t_t\sigma_{ij}$) could be expressed by

$$t_i^{\mathrm{eng}} = {}^t_0t_i = {}^t_0\sigma_{ij}\ {}^0e_j \qquad \text{and} \qquad t_i^{\mathrm{true}} = {}^t_tt_i = {}^t_t\sigma_{ij}\ {}^te_j .$$

For clarity we introduce the following self-explaining notations

$$\sigma_{ij}^{\mathrm{eng}} = {}^t_0\sigma_{ij} \qquad \text{and} \qquad \sigma_{ij}^{\mathrm{true}} = {}^t_t\sigma_{ij} .$$

The advantage of using the engineering stress is evident. Its calculation simply employs the body dimensions of non-deformed configuration and the consequent errors – due to small displacements and strains – are acceptable. The engineering stress, however, is useless for nonlinear tasks where finite strains and rigid body rotations cannot be neglected.

The conception and definition of true stress are quite clear, but its direct evaluation is impossible since the true stress depends on unknown geometry of the deformed configuration, which is result of the applied force which caused that deformation.

It is, however, the true (Cauchy) stress, which we, the engineers, are only interested in. Often we are satisfied with its engineering approximation. Since, however, we are not able to evaluate the true stress we have to employ a trick invented by forefathers of continuum mechanics two centuries ago. Namely to solve matter indirectly, introducing various fictitious stress measures, as for example the first and the second Piola-Kirchhoff stress tensors. These stress measures are defined by means of fictitious forces, acting in ${}^{0}C$ configuration, that are systematically derived from those actually acting in the deformed configuration ${}^{t}C$ (i.e. forces that are responsible for the deformation) with the intention that the invented new measures would be independent of coordinate system orientation and furthermore insensitive to rigid body motions.

The procedure for deriving the above mentioned fictitious measures is based on the conservation mass theorem and on kinematic relations describing the deformation ${}^{0}C \to {}^{t}C$. If the Lagrangian formulation is used then the transformation process (i.e. the deformation) is defined by a function ${}^{t}x_i = {}^{t}x_i({}^{0}x_j, t)$ and if the Jacobian of the transformation, say $J = \det F_{ij} \neq 0$, with F_{ij} being the deformation gradient defined by $F_{ij} = \dfrac{\partial\, {}^{t}x_i}{\partial\, {}^{0}x_j}$, then one could express the 'deformed' elementary line, say $\mathrm{d}^{t}x_i$, in ${}^{t}C$, as a function of the same line, say $\mathrm{d}^{0}x_i$, in the reference configuration ${}^{0}C$ by means of $\mathrm{d}^{t}x_i = {}^{t}_{0}F_{ij}\, \mathrm{d}^{0}x_j$.

Based on requirements (invariance to coordinate system orientation and rigid body motions) and using the above relations, a new measure – expressed in reference configuration – called the first Piola-Kirchhoff stress tensor and denoted ${}^{t}_{0}\sigma_{ij}^{\mathrm{1PK}}$ here, can be derived as a function of true stress. Without going to details, that can be found elsewhere, one can write

$${}^{t}_{0}\boldsymbol{\sigma}^{\mathrm{1PK}} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{t}_{0}\mathbf{F}\, {}^{t}_{t}\boldsymbol{\sigma}^{\mathrm{true}} .$$

This stress measure is not symmetric. To achieve the symmetry, being usual for other stress measures, a similar measure (called the second Piola-Kirchhoff stress tensor) is derived applying the relation between geometric quantities to applied forces as well. One then gets

$${}^{t}_{0}\boldsymbol{\sigma}^{\mathrm{2PK}} = \frac{{}^{0}\rho}{{}^{t}\rho}\, {}^{t}_{0}\mathbf{F}^{-1}\, {}^{t}_{t}\boldsymbol{\sigma}^{\mathrm{true}}\, {}^{t}_{0}\mathbf{F}^{-\mathrm{T}} .$$

Even if the shown stress tensors have no physical meaning and cannot in any way be measured, they became useful tools for solving the nonlinear task of continuum mechanics, especially in cases where large strains and/or rotations prevails. Still, there is a hint of reality in them – in case of small strains and rotations the first and second Piola-Kirchhoff stresses approach the true stress which in turn tends towards the engineering stress. The stress measures require to be coupled with suitable energetically conjugate strain measures. In case of the second Piola-Kirchhoff stress tensor it is the Green-Lagrange strain tensor that should be employed. But this is another story.

For more details see [3].

5. Stress rates and conjugate stress and strain measures

As mentioned before the stress measures mean nothing without energetically conjugate strain measures that are associated to them. This necessitates in expressing stress measures by means of their increments ($\Delta_0^t\boldsymbol{\sigma}^{2\mathrm{PK}}$) instead of measures themselves, i.e $_0^t\boldsymbol{\sigma}^{2\mathrm{PK}}$. The stress rates are instantaneous values of stress measures. For example the second Piola-Kirchhoff stress rate tensor is defined by $_0^t\dot{\boldsymbol{\sigma}}^{2\mathrm{PK}} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta_0^t\boldsymbol{\sigma}^{2\mathrm{PK}}}{\Delta t}$. The energetically conjugate strain rate in this case is the Green-Lagrange strain tensor. The tress rates are useful in efficient formulations of incremental forms of governing equations needed for expressing iterative processes required for reaching the sought after equilibrium conditions.

The governing equations in rate forms are more elegant than the incremental ones, but when it comes to their algorithmization one have express the instantaneous values by increments since in a computer treatment of mathematic formulas we are not able to express a derivative of a function. It always has to be approximated numerically by increments.

It should be reminded that the true stress rate cannot be directly used in iterative rate formulations since it is not, being dependent on coordinate system orientation, objective. That's there is plethora of other stress rates (and corresponding strain rates) defined in continuum mechanics literature and commonly used in commercial finite element packages. The Jaumann and Green-Naghdi stress rates are good examples. For more details see [4].

The stress rates and properly chosen energetically conjugate strain rates present an interesting topic that is still under discussion in continuum mechanics community. The subject is rather tricky and is not yet fully understood by engineering community as documented by [5]. Another interesting paper devoted to this subject can be found in [6].

6. Conclusions

When mechanical quantities being defined by consent – as force, stress, energy, etc. – are treated a lot of assumptions are a priory accepted regardless whether analytical, numerical or/and experimental approaches are employed. This requires pondering a little bit about the proper meaning of terms and about basic definitions of mechanical variables we are using.

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